We introduce a time-dependent climate feedback parameter \( \lambda(t) \) in Budyko’s (1969) linear relationship between surface temperature and outgoing long-wave radiation. Using two different methods, we derive an energy balance model (EBM) with \( \lambda(t) \) from energy conservation principles applied to the climate system. We show the EBM with a time-variable \( \lambda(t) \) from energy conservation principles applied to the climate system.

The theoretical development is as follows:

**Method I: Partial derivatives**

We assume \( \Delta R \) is small enough to be considered a linear perturbation. Following this, we develop the total derivative formula:

\[
\Delta R = \lambda_0 \Delta T_S + T_{S0} \Delta \lambda(t) - \lambda(t) \Delta T_S(t) + \lambda_0 \Delta T_S(t) + T_{S0} \Delta \lambda(t) - \lambda(t) \Delta T_S(t)
\]

\[
C \frac{d \Delta T_S}{dt} = \Delta F + T_{S0} \Delta \lambda(t) - \lambda(t) \Delta T_S(t)
\]

**Method II: Perturbation theory**

We assume a small deviation from steady state \( T_{S0} \) and \( \lambda_0 \) for simplicity. Ignoring the perturbed dynamical system, the total derivative follows:

\[
\Delta R = \lambda_0 \Delta T_S + T_{S0} \Delta \lambda(t) - \lambda(t) \Delta T_S(t) + \lambda_0 \Delta T_S(t) + T_{S0} \Delta \lambda(t) - \lambda(t) \Delta T_S(t)
\]

Perturbation theory states that a solution to this system can be found close to a solution of the unperturbed system.

\[
\frac{d \Delta T_S}{dt} = \frac{\Delta F}{C} + \frac{T_{S0} \Delta \lambda(t)}{C} - \frac{\lambda(t) \Delta T_S(t)}{C}
\]

We look for a 1st-order solution to the budget equation with \( T_{S0} = T_{S0} + \Delta T_S \) and \( \lambda = \lambda_0 + \lambda(t) \).

**Theoretical conclusion**

\[
\Delta R(t) = \Delta T_S(t) + T_{S0} \Delta \lambda(t) - \lambda(t) \Delta T_S(t)
\]

The surface temperature anomaly follows:

\[
\frac{d \Delta T_S}{dt} = \Delta F + T_{S0} \lambda_0 + \lambda(t) \Delta T_S(t)
\]

**Validation and Consequences**

- **Reproducing the surface temperature dynamics**
  - Numerical integration with 3-layer ocean
  - Global average surface temperature (K)
  - Temperature distance from ESM output (K)

- **Comparing time varying \( \lambda \) in abrupt CO\(_2\) increase experiments**
  - In the literature
  - Many definitions of time varying \( \lambda \) exist
  - \( \lambda \) is defined using \( T_{S0} \) and \( \lambda_0 \)

- **The constant temperature experiment**
  - Test if \( \Delta R = \lambda(t) \Delta T_S(t) \) is true using a constant temperature experiment, with varying \( \lambda(t) \) to include the pattern effect.

**Results**

- HadGEM3-ESM
- MRI-ESM2
- CESM2
- MIROC6
- IPSL-CM6A
- CanESM5

- The usual \( \Delta R \) leads to absurd values of \( \lambda(t) \) and requires the supplementary term \( \Delta \lambda(t) \)

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