On the relationship between PI and CAPE

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1. Approximations in the CAPE algorithm*
2. Approximations in the closed formulas
3. Updates to the closed formulas

*Bister and Emanuel JGR 2002
CAPE defined as a contour integral

\[ CAPE = \int_{p_0}^{p_b} (\alpha - \alpha_0) dp = -\oint_{\Gamma} \alpha dp \]

“Hurricane CAPE” may have \( \alpha \neq \alpha_0 \) at bottom
Vertical cross-section in pressure coordinates

Dynamical constraint:

$$-\alpha dp = d(v^2/2 + \Phi)$$
\[ \Gamma \quad - \oint_{\Gamma} \alpha dp = -\frac{1}{2} (v^2 - v'^2) + R T_{vb} \log \left( \frac{p_e}{p} \right) \]

\[ \Gamma_o \quad - \oint_{\Gamma_o} \alpha dp = CAPE \]

\[ \Gamma_b \quad - \oint_{\Gamma_b} \alpha dp = R (T_{vb} - T_{vo}) \log \left( \frac{p_e}{p} \right) \]

\[ R T_{v_0} \log \left( \frac{p_e}{p} \right) = CAPE + \frac{1}{2} (v^2 - v'^2) \]

Holland 97
Bister-Emanuel 02

GFDL/NOAA
If SLP is known at a second overturning radius (a)

\[ RT_{v_0} \log \left( \frac{p_a}{p_c} \right) = CAPE_c - CAPE_a + \frac{1}{2} (v_c^2 - v_a^2) \]
\[ RT_{v_0} \log \left( \frac{p_a}{p_c} \right) = CAPE_c - CAPE_a \]

Take \( p_a = p_c \); solve for \( p_c \)

Pressure PI with \( \nu^2 = 0 \)

\[ RT_{v_0} \log \left( \frac{p_e}{p} \right) = CAPE \]

Solve for \( p \)
Pressure PI with $\nu^2 \neq 0$

\[ p_a = p_c \] (code)

\[ p_a < p_c \] (BE 2002)
## Definitions of PI

<table>
<thead>
<tr>
<th>Condition</th>
<th>IEW ((\nu^2 = 0))</th>
<th>OEW ((\nu^2 &gt; 0))</th>
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<tbody>
<tr>
<td>(p_a = p_e)</td>
<td>E86, E88</td>
<td>E95, code</td>
</tr>
<tr>
<td>(p_a &lt; p_e)</td>
<td>H97</td>
<td>BE02</td>
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</table>
Velocity closure

\[-\alpha dp = T ds - dk\]

\[-\frac{1}{2}(v^2 - v'^2) + RT_{vb} \log \left( \frac{p_e}{p} \right) = (\bar{T}_b - \bar{T}_o)(s - s_e)\]

Differentiate once wrt inflow radius:

\[-\left( \frac{1}{r^2} - \frac{1}{r_e^2} \right) M \frac{\partial M}{\partial r} = (T_b - T_o) \frac{\partial s}{\partial r}\]

Then use aerodynamic law:

\[v^2 = \frac{C_k}{C_d} (T_b - T_o)(s^* - s)\]
$\text{CAPE}^* = \int_{T_0}^{\bar{T}_B} \alpha dp = \int_{T_0}^{T_0^*} \alpha dp = (\bar{T}_B - T_0)(s^* - s_0(p^*))$

$\bar{T}_B = [k^* - k_0(p^*)]/s^* - s_0(p^*)$
\[ \text{CAPE}^* - \text{CAPE} = (\tilde{T}_B - \tilde{T}_o)(s^* - s) \]

\[ \tilde{T}_B = (k^* - k)/(s^* - s) \]

\[ \tilde{T}_o = \int_s^{s^*} T_0 ds / (s^* - s) \]

Compare:

\[ v^2 = \frac{c_k}{c_d} (T_b - T_o)(s^* - s) \]

\[ \Rightarrow v^2 \approx \frac{c_k}{c_d} (\text{CAPE}^* - \text{CAPE}) \]

Bister-Emanuel 02
Error from $T$ averaging (%)

$\tilde{T}_b - \tilde{T}_o \quad \text{vs} \quad T_b - T_o$

relative error due to averaging (revers)

relative error due to averaging (pseudo)
2. Approximations in the thermodynamics: Fully moist vs. “moist-like”

“Moist-like” neglects condensate heat capacity, loading
IEW pressure PI (hPa, pseudo)

Pressure PI with $\nu^2 = 0$

IEW pressure PI (hPa, dry-pseudo)
Full moisture (code)

Velocity PI

Moist-like

velocity PI (m/s, pseudo)

velocity PI (m/s, dry–pseudo)
3. Closed formulas for full moisture

Thermodynamics:

\[
(c_p + w c_{pv} + l c_l) dT - (1 + Q) \alpha dp = -L dw \quad 1^{\text{st}} \text{ law}
\]

\[-\alpha dp = T ds - dk + Q \alpha dp + c_l T dQ \quad 1^{\text{st}} \& 2^{\text{nd}} \text{ laws}
\]

Dynamics:

\[-\alpha dp = d(v^2/2 + \Phi)\]

\[
k = (c_p + c_l Q) T + Lw
\]

\[
Q = w + l
\]

\[
\Phi = gz
\]
Irreversible case (full moisture)

Define:

\[\int_0^{\Phi_o} Q d\Phi = w_b \Phi_q\]
\[\int_{T_b}^{T_0} Q dT = w_b (T_q - T_b)\]
\[\int_{0T_b}^{T_0} Q d\log T = w_b (\Lambda_q - \Lambda_b)\]

\[
\frac{(1 + w_b)RT_{vb}}{p_c} \log \left( \frac{p_a}{p_c} \right) = D_{a,c} + (1 + \bar{w}_c) \frac{v_c^2}{2} - (1 + \bar{w}_a) \frac{v_a^2}{2}
\]

\[D_{a,c} = (\bar{T}_b - \bar{T}_c)(s_c - s_a) + [(G_b - G_q)w_b]_a^c\]

\[G = \Phi + c_i(T - \bar{T}_c \log T)\]
Velocity PI for irreversible full moisture case

\[ \nu^2 = \frac{c_k}{c_a} (1 + w)^{-1} \left[ (T_b - T_o)(s - s) + (G_b - G_q)(w^* - w) \right] \]
Effective outflow geopotential $\Phi_q$ (pseudoadiabatic)