Stability and accuracy of Runge–Kutta-based split-explicit time-stepping algorithms for free-surface ocean models

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OMDP/COMMODORE workshop, Sept 2024, Boulder

Context

- As part of the redesign of the NEMO GCM time integration algorithm, a Leapfrog-Robert/Asselin—based split-explicit algorithm

- We consider the feasibility of building reliable algorithms based on simple RK schemes for the 3D part :

'RK3' : $d_t\phi = f(\phi)$ $\longrightarrow$

$$\begin{aligned} \phi^{n+1/3} &= \phi^n + \frac{\Delta t}{3} f(\phi^n), \\ \phi^{n+1/2} &= \phi^n + \frac{\Delta t}{2} f(\phi^{n+1/3}), \\ \phi^{n+1} &= \phi^n + \Delta t f(\phi^{n+1/2}), \end{aligned}$$

Wicker & Skamarock 2002 (WRF, MPAS, …)

$$\mu = \omega \Delta t \leq \mu^{\star}_{\mathrm{RK3}} = \sqrt{3}$$

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- We consider the feasibility of building reliable algorithms based on simple RK schemes for the 3D part :

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$$\phi^{n+1/3} = \phi^n + \frac{\Delta t}{3} f(\phi^n),$$
$$\phi^{n+1/2} = \phi^n + \frac{\Delta t}{2} f(\phi^{n+1/3}),$$
$$\phi^{n+1} = \phi^n + \Delta t f(\phi^{n+1/2}),$$

How sensitive are Split-RK3 algorithms to the way the 2D/3D coupling is arranged within the three stages?

Are there any arrangements that provide better stability and/or accuracy? Which ones?

In this talk

- I introduce the different 2D/3D coupling strategies we've studied,
- I summarize the results of a stability/accuracy analysis of Split-RK3 algorithms,

Journal of Computational Physics 398 (2019) 108875

Stability analysis of split-explicit free surface ocean models: Implication of the depth-independent barotropic mode approximation

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→ A framework for split-explicit algorithm analysis based on vertical modes decomposition

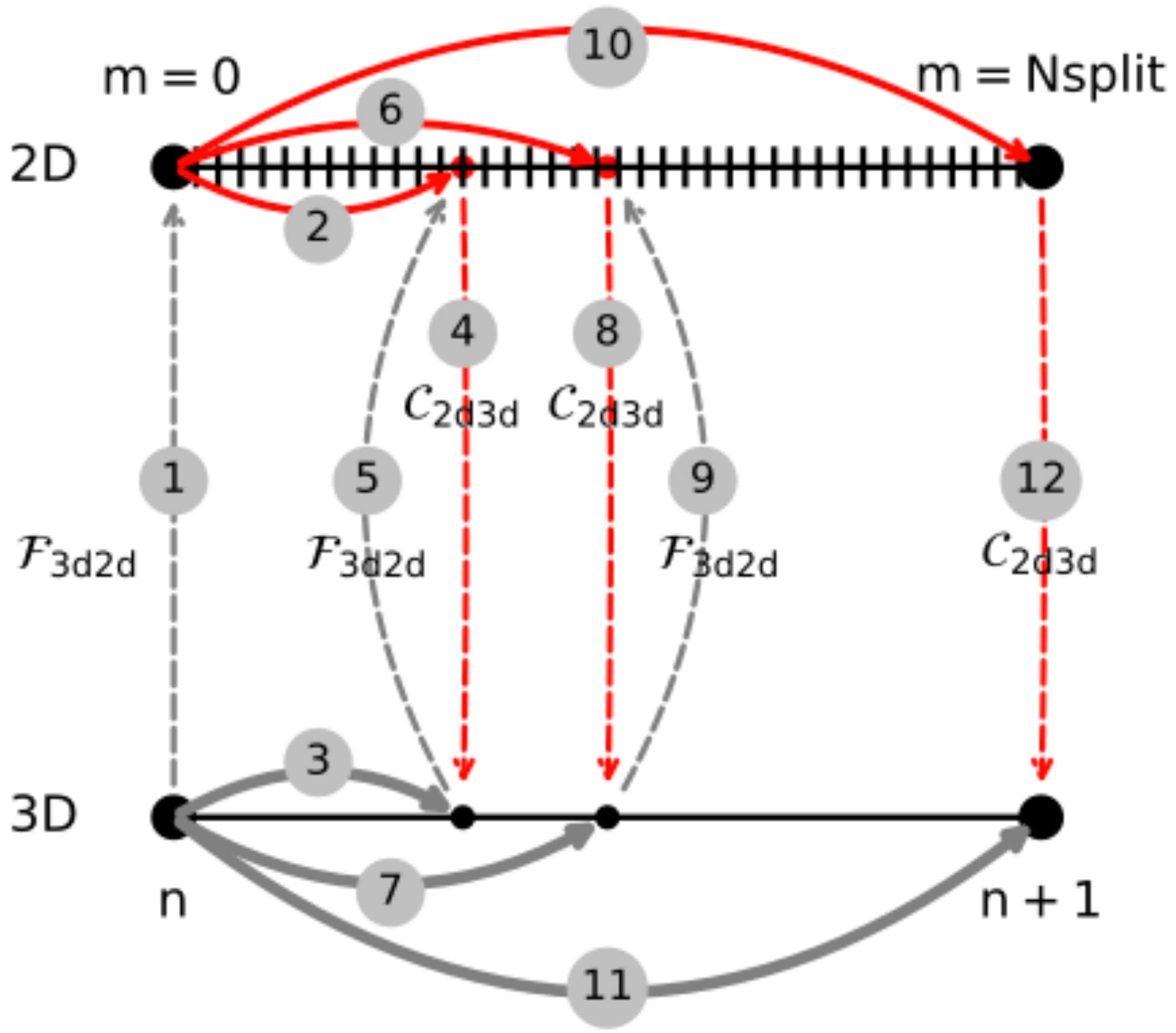
→ Counteract the effects of splitting errors by using dissipative schemes for the 2D integration, with tunable damping, rather than using averaging filters.

- I report on a few numerical experiments to help verify the results of the analysis.

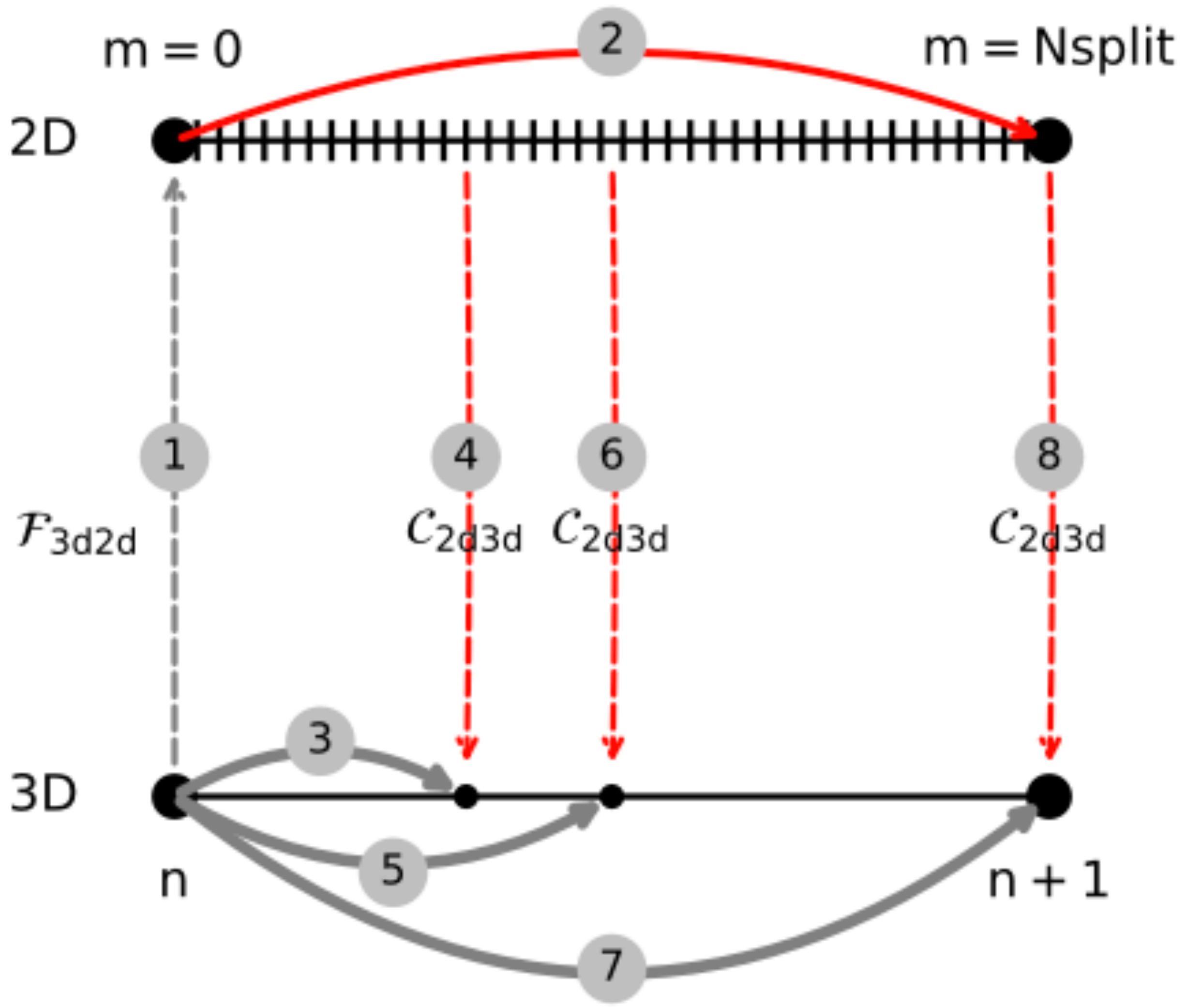
1 - The 2D/3D coupling strategies we have considered

With RK3 as the 3D scheme :

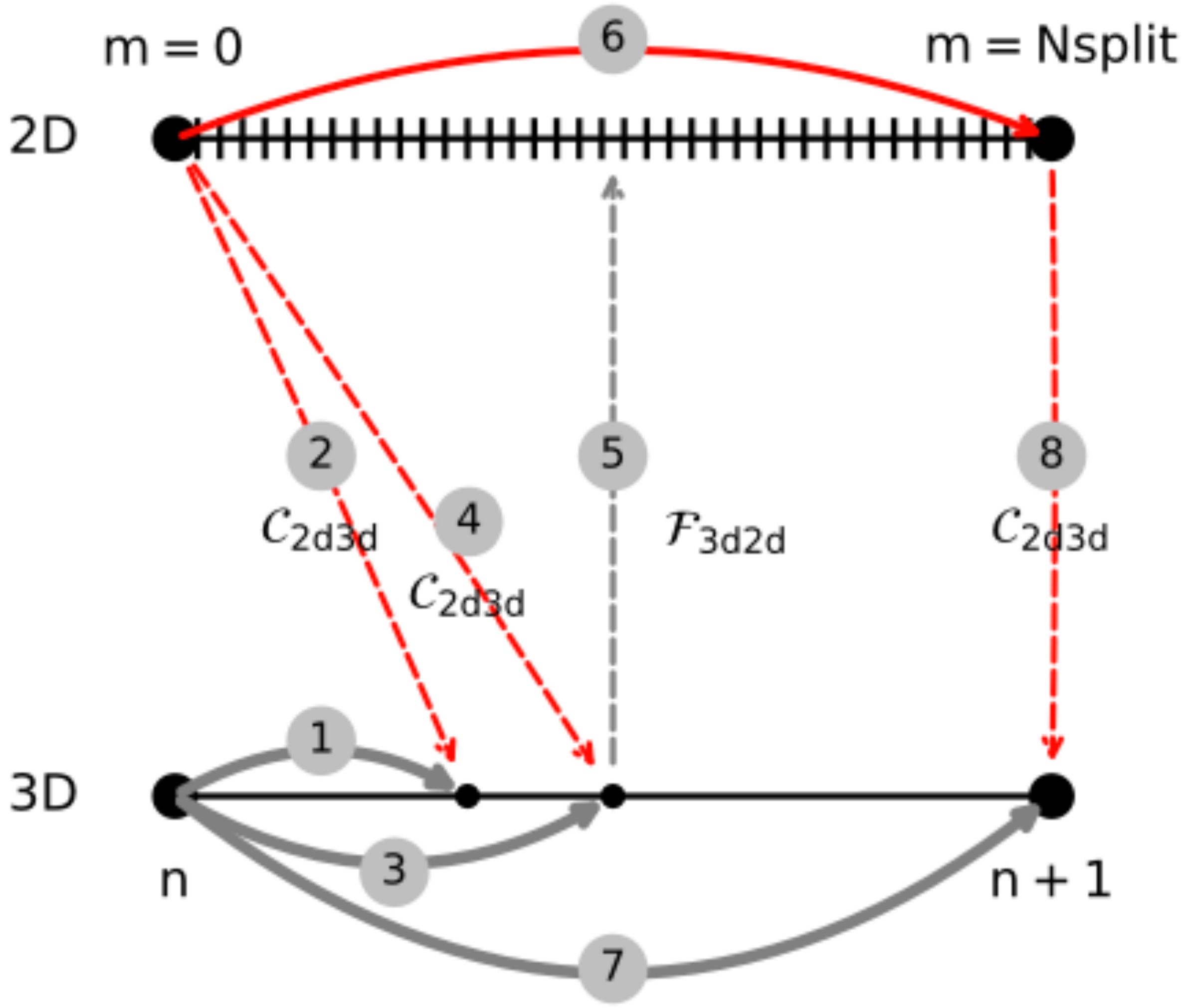
Each-stage strategy



First-stage strategy



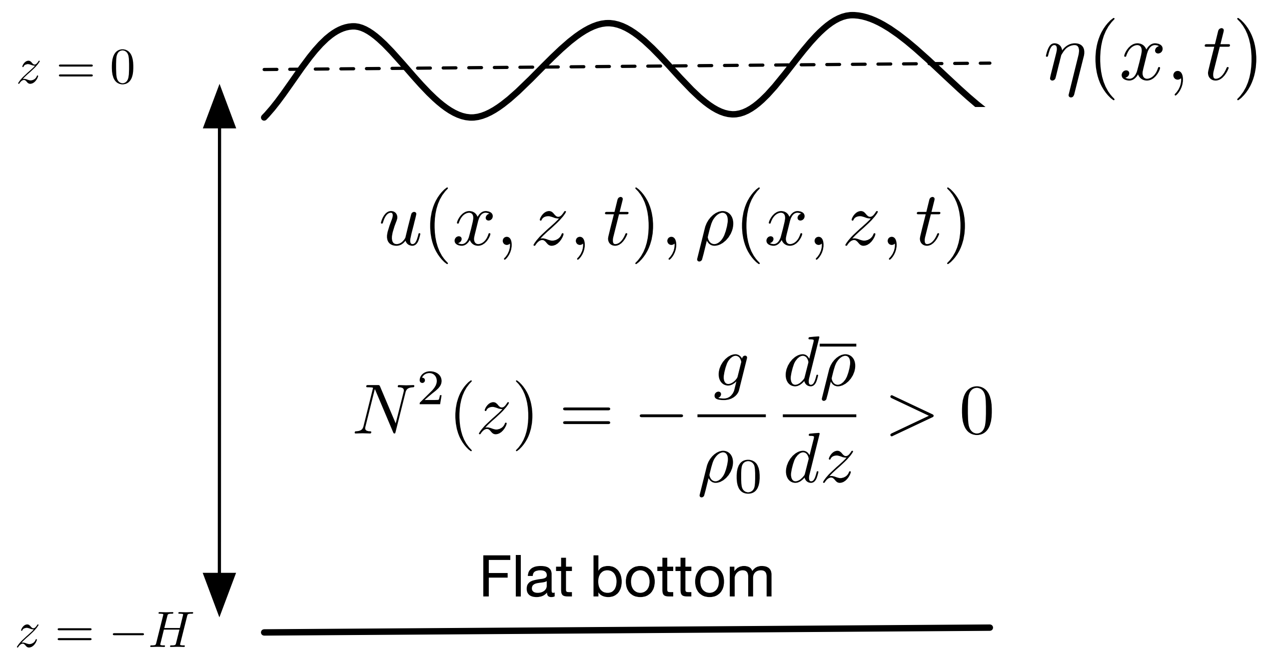
Last-stage strategy



2 - Linear stability/accuracy analysis

The framework proposed by Demange et al. 2019

- The x-z inviscid, adiabatic HPEs linearized about rest, without rotation :



$$
\begin{cases}
\partial_t u + \frac{1}{\rho_0}\partial_x p &= 0 \\
\partial_z p &= -\rho g \\
\partial_x u + \partial_z w &= 0 \\
\partial_t \rho + w\frac{d\overline{\rho}}{dz} &= 0
\end{cases}
$$

with the surface bc

$$
\begin{aligned}
w(z=0) &= \partial_t \eta \\
p(z=0) &= \rho_0 g \eta
\end{aligned}
$$

and the bottom bc

$$
w(z=-H) = 0
$$

2 - Linear stability/accuracy analysis

The framework proposed by Demange et al. 2019

- The x-z inviscid, adiabatic HPEs linearized about rest, without rotation :
- The vertical normal mode decomposition :
  - A stratification-related Sturm-Liouville problem : $M_q(z),\ \lambda_q(=1/c_q^2)$
  - An equivalent formulation of HPEs :

$$u_q = \langle u, M_q \rangle \qquad h_q = \frac{1}{\rho_0 g}\langle p, M_q \rangle \quad \longrightarrow \quad \partial_t u_q = -g\,\partial_x h_q \qquad \partial_t h_q = -(c_q^2/g)\,\partial_x u_q$$

  - For constant stratification :

$$\varepsilon = \frac{N^2 H}{g} = \mathcal{O}(10^{-2} - 10^{-4})$$

$$M_0(z) = 1 - \varepsilon\left[\frac{1}{3} + \frac{z}{H} + \frac{z^2}{2H^2}\right] + \mathcal{O}(\varepsilon^2), \quad c_0 = \alpha_0\sqrt{gH} \text{ with } \alpha_0 = 1 + \frac{\varepsilon}{6} + \mathcal{O}(\varepsilon^2),$$

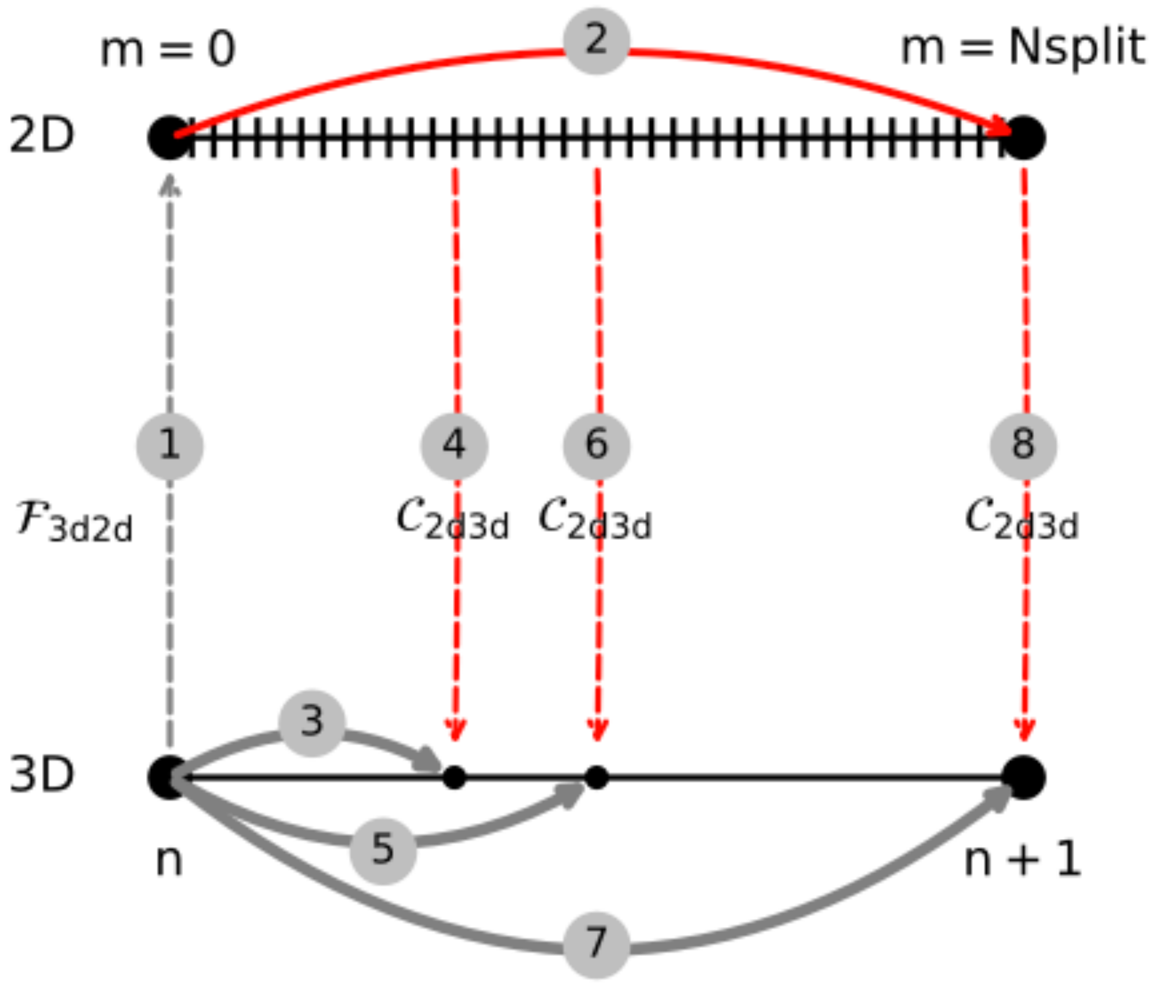
$$M_q(z) = \sqrt{2}\left(\cos\left(\frac{q\pi z}{H}\right) - \frac{\varepsilon}{q\pi}\sin\left(\frac{q\pi z}{H}\right)\right) + \mathcal{O}(\varepsilon^2), \quad c_q = \alpha_q\sqrt{gH} \text{ with } \alpha_q = \frac{\sqrt{\varepsilon}}{q\pi} + \mathcal{O}(\varepsilon^{3/2})$$

2 - Linear stability/accuracy analysis

The framework proposed by Demange et al. 2019

- The x-z inviscid, adiabatic HPEs linearized about rest, without rotation :
- The vertical normal mode decomposition :
- Formulating split-explicit algorithms in terms of modal projections : $\mathbf{x}_q = (u_q, h_q)^T$

First-stage strategy



1^{st} stage		
1 & 2	$\overline{\mathbf{x}}^{n+1} = \left[\mathbf{A}^{2d}\right]^{N_{\text{split}}} \sum_p \mathbf{V}_p \mathbf{x}_p^n + \left(\left[\mathbf{A}^{2d}\right]^{N_{\text{split}}} - \mathbf{I}\right) \sum_p \mathbf{Q}_p \mathbf{x}_p^n$	2D integration
3	$\mathbf{x}_q^{n+1/3} = \mathbf{x}_q^n + \frac{1}{3}\mathbf{A}_q \mathbf{x}_q^n$	3D integration
4	$\mathbf{x}_q^{n+1/3,c} = \mathbf{x}_q^{n+1/3} + \mathbf{C}_q\left(\overline{\mathbf{x}}^{n+1/3} - \sum_p \mathbf{V}_p \mathbf{x}_p^{n+1/3}\right)$	2D/3D correction
	with $\overline{\mathbf{x}}^{n+1/3} = \gamma_1 \overline{\mathbf{x}}^n + (1-\gamma_1)\overline{\mathbf{x}}^{n+1}$	
2^{nd} stage		
5	$\mathbf{x}_q^{n+1/2} = \mathbf{x}_q^n + \frac{1}{2}\mathbf{A}_q \mathbf{x}_q^{n+1/3,c}$	3D integration
6	$\mathbf{x}_q^{n+1/2,c} = \mathbf{x}_q^{n+1/2} + \mathbf{C}_q\left(\overline{\mathbf{x}}^{n+1/2} - \sum_p \mathbf{V}_p \mathbf{x}_p^{n+1/2}\right)$	2D/3D correction
	with $\overline{\mathbf{x}}^{n+1/2} = \gamma_2 \overline{\mathbf{x}}^n + (1-\gamma_2)\overline{\mathbf{x}}^{n+1}$	
3^{rd} stage		
7	$\mathbf{x}_q^{n+1} = \mathbf{x}_q^n + \mathbf{A}_q \mathbf{x}_q^{n+1/2,c}$	3D integration
8	$\mathbf{x}_q^{n+1,c} = \mathbf{x}_q^{n+1} + \mathbf{C}_q\left(\overline{\mathbf{x}}^{n+1} - \sum_p \mathbf{V}_p \mathbf{x}_p^{n+1}\right)$	2D/3D correction

2 - Linear stability/accuracy analysis

The framework proposed by Demange et al. 2019

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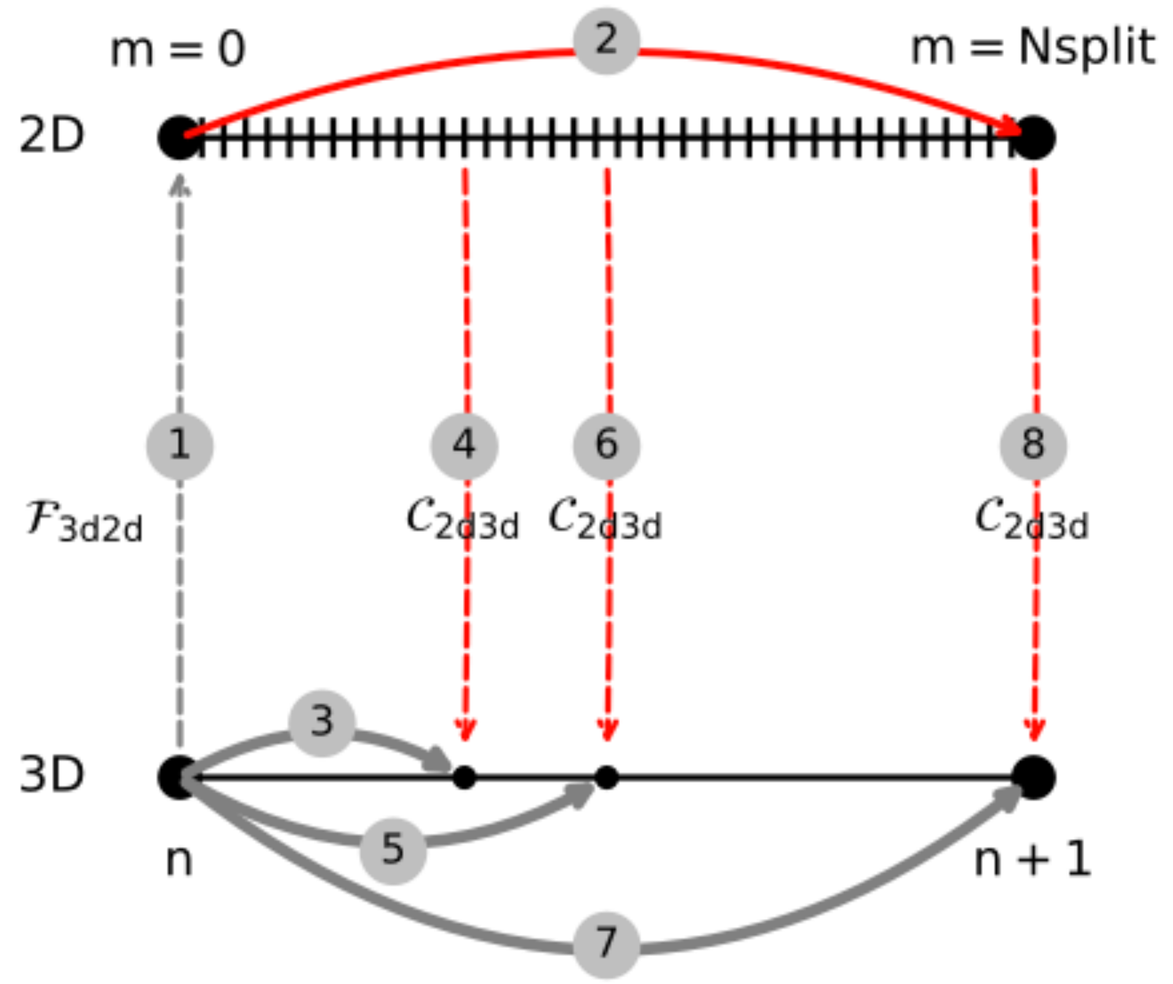
First-stage strategy



→

Build the step-multiplier matrix

$$\mathbf{x}_q^{n+1,c} = \mathbf{G}_q(\mathbf{x}_q^n, \mathbf{x}_p^n) \text{ for } q < N_{\text{modes}}$$

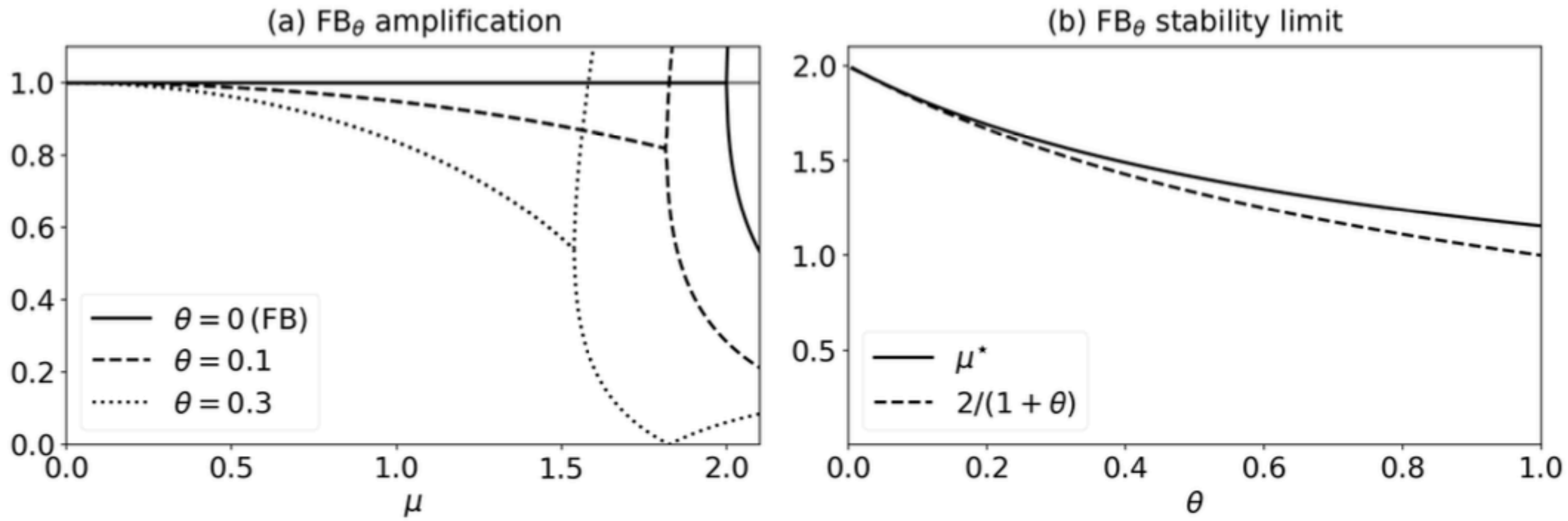
then solve for eigenvalues as a function of $\mu_0 = c_0 k \Delta t$

2 - Linear stability/accuracy analysis

Damping using dissipative 2D time-stepping schemes

- e.g. a dissipative Forward-Backward

$$\overline{u}^{m+1} = \overline{u}^m - g\Delta t_0 \partial_x \eta^m$$
$$\eta^{m+1} = \eta^m - H\Delta t_0 \partial_x \left((1+\theta)\overline{u}^{m+1} - \theta \overline{u}^m\right)$$



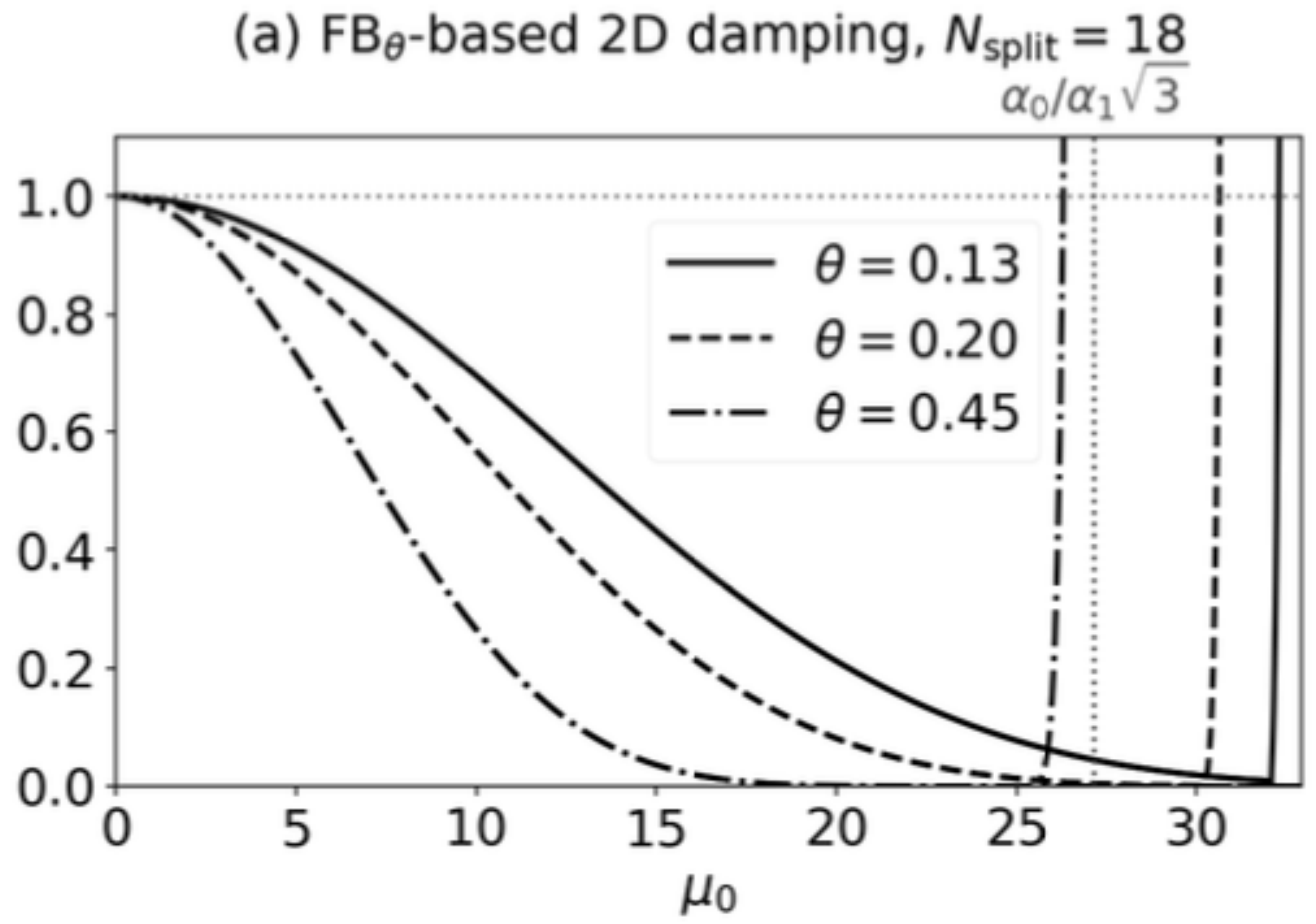
2 - Linear stability/accuracy analysis

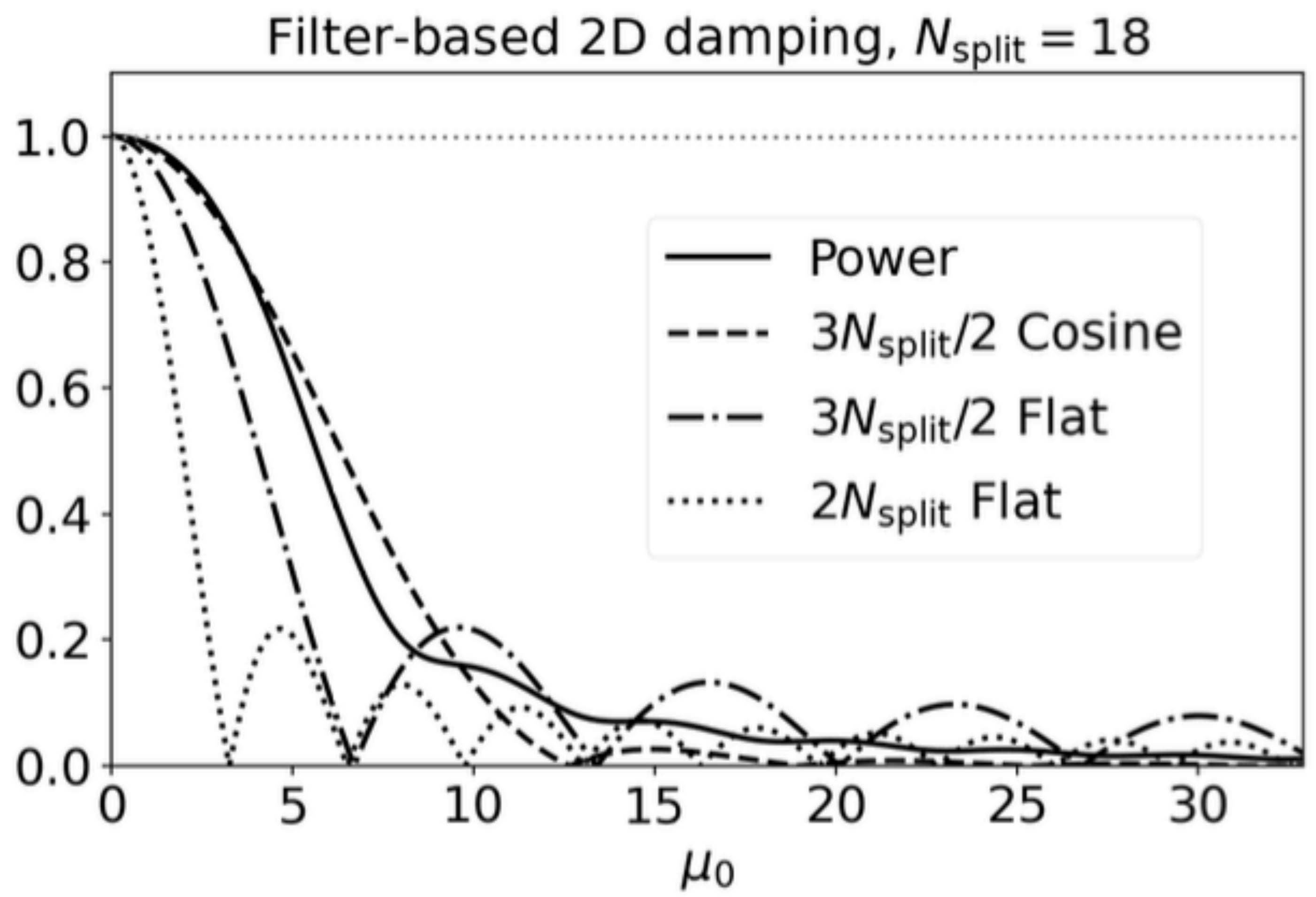
Damping using dissipative 2D time-stepping schemes

- e.g. a dissipative Forward-Backward
- when used in split-explicit algorithms

$$N = 10^{-2}\,\mathrm{s}^{-1},\ H = 4000\,\mathrm{m},\ g = 9.81\,\mathrm{m.s}^{-2}$$

$$\longrightarrow \varepsilon = \frac{N^2 H}{g} \approx 0.041 \quad \frac{c_0}{c_1} \approx 15.6$$





→ a tunable, 2nd-order at large scales, damping

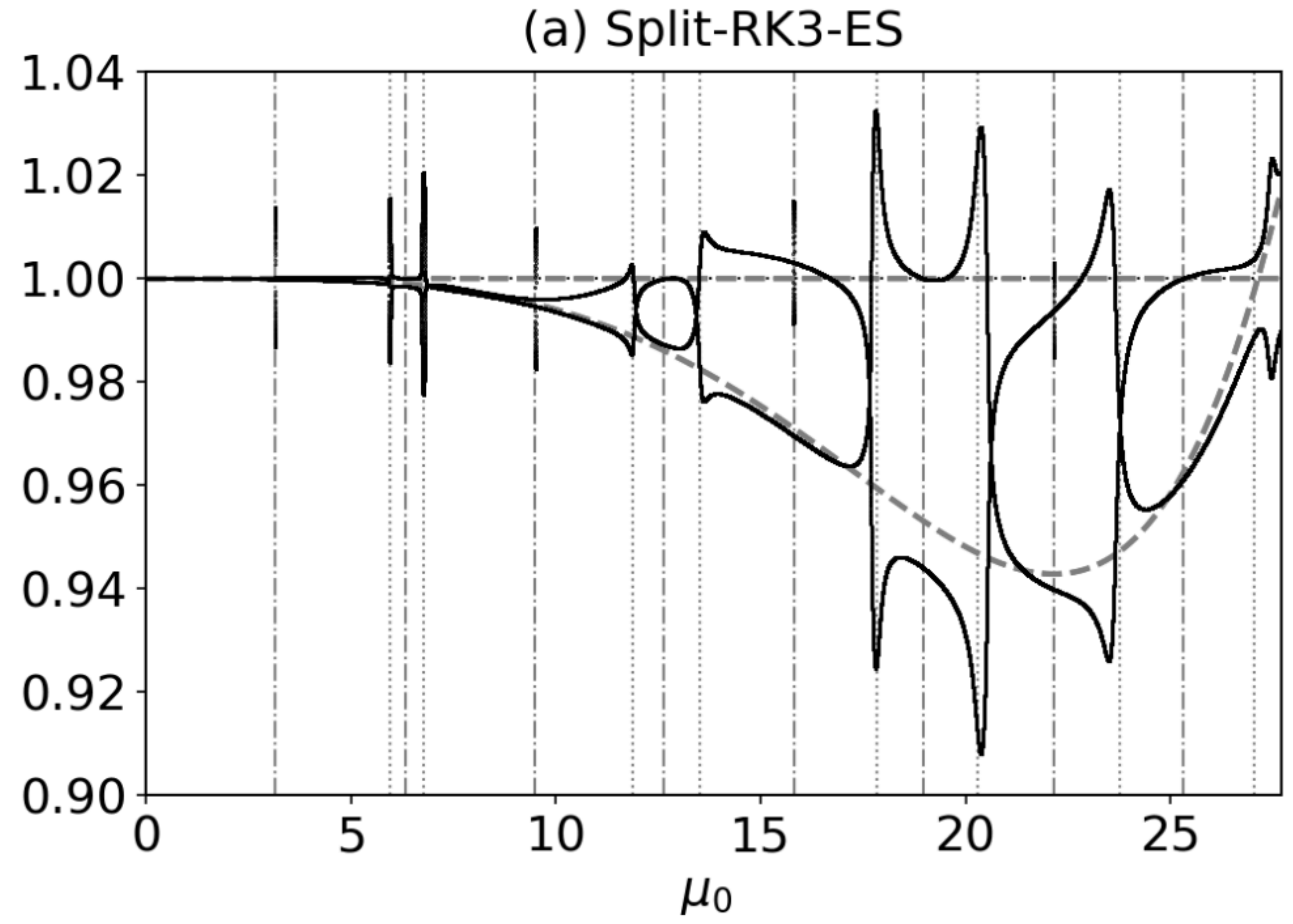
→ able to stabilise the Split-RK3 algorithms?

2 - Linear stability/accuracy analysis

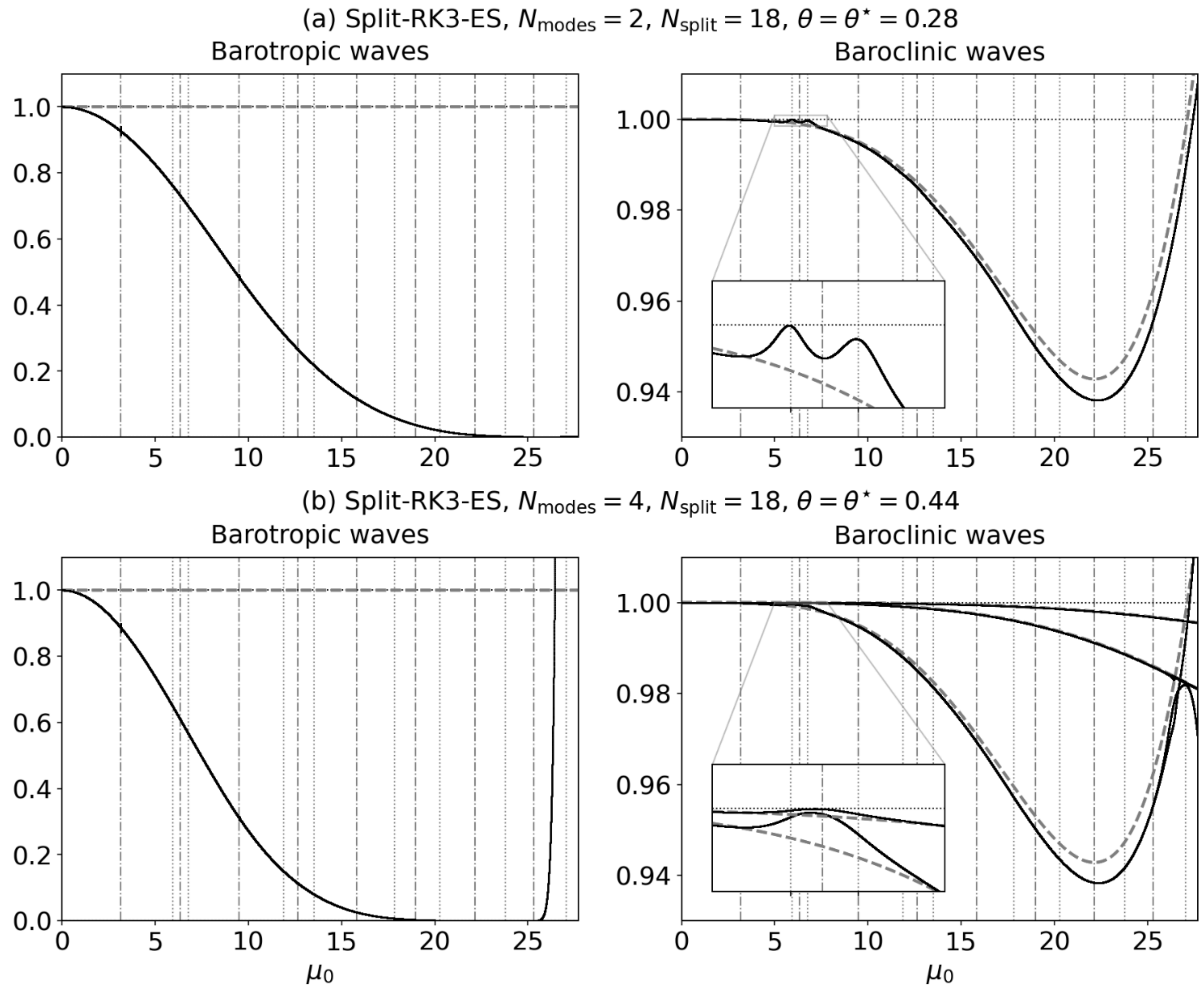
The results : Split-RK3 with the Each-Stage strategy

$$\varepsilon = \frac{N^2 H}{g} \approx 0.041 \quad \frac{c_0}{c_1} \approx 15.6$$

Without 2D damping



With 2D damping



→ Split-RK3-ES is difficult to stabilise as the dominant instabilities involve higher-modes baroclinic waves, are weakly controlled by the 2D damping

→ We discard this strategy

2 - Linear stability/accuracy analysis

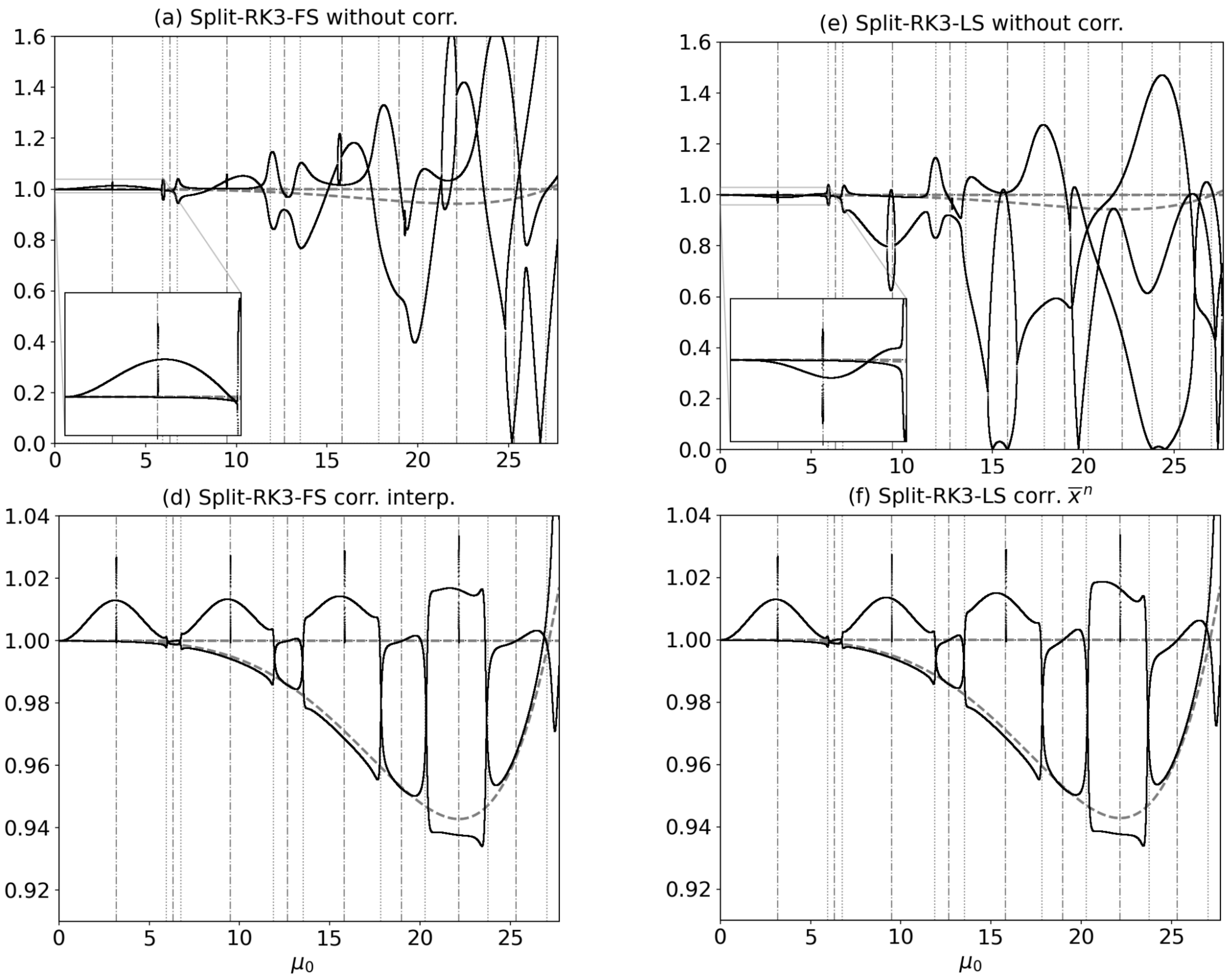
The results : Split-RK3 with the First-Stage and Last-stage strategies

$$\varepsilon = \frac{N^2 H}{g} \approx 0.041 \quad \frac{c_0}{c_1} \approx 15.6$$

Without 2D damping
& Without intermediate corrections

Without 2D damping
& With intermediate corrections

→ Applying corrections at intermediate RK3 stages greatly reduces the instabilities at all scales

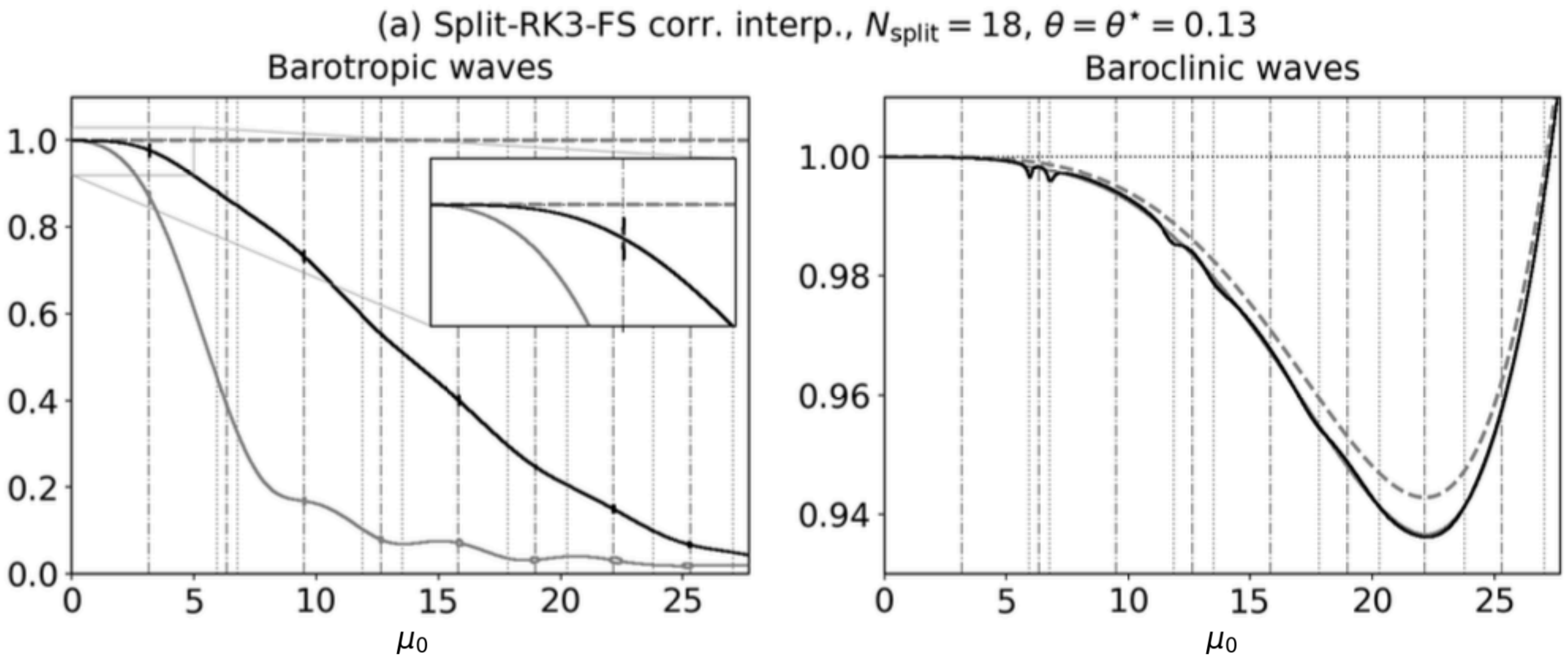


2 - Linear stability/accuracy analysis

The results : Split-RK3 with the First-Stage and Last-stage strategies

$\varepsilon = \frac{N^2 H}{g} \approx 0.041 \quad \frac{c_0}{c_1} \approx 15.6$

With 2D damping, with corrections



$$\frac{\theta}{N_{\text{split}}} \geq \frac{\varepsilon}{6}$$

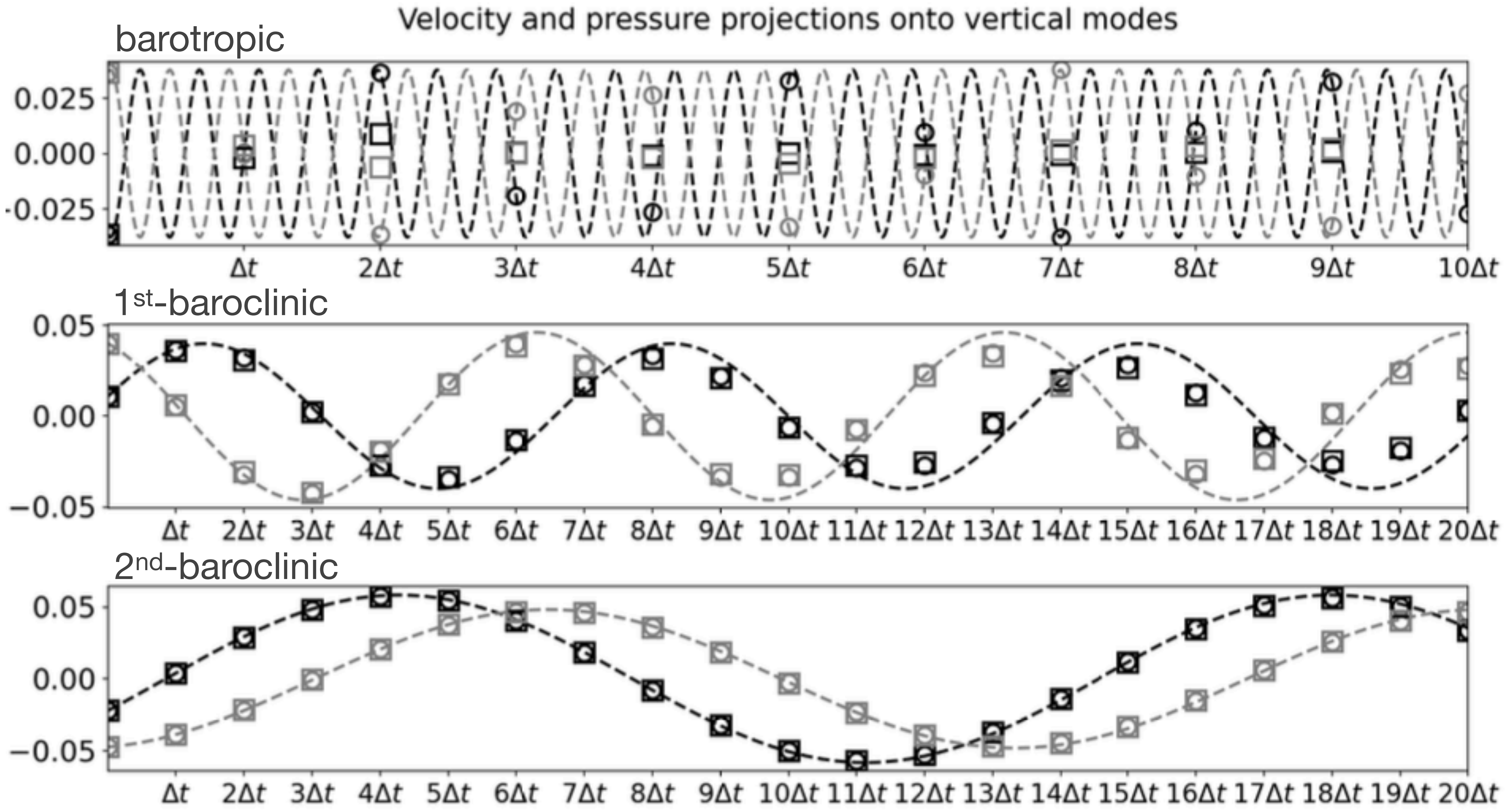
- → Split-RK3-FS/LS, with intermediate corrections, can be robustly stabilised as the dominant instabilities are barotropic at large scales and efficiently controlled by the 2D damping
- → A useful condition for the minimum 2D damping required for stability
- → A first-order in time algorithm, for both barotropic and baroclinic dynamics

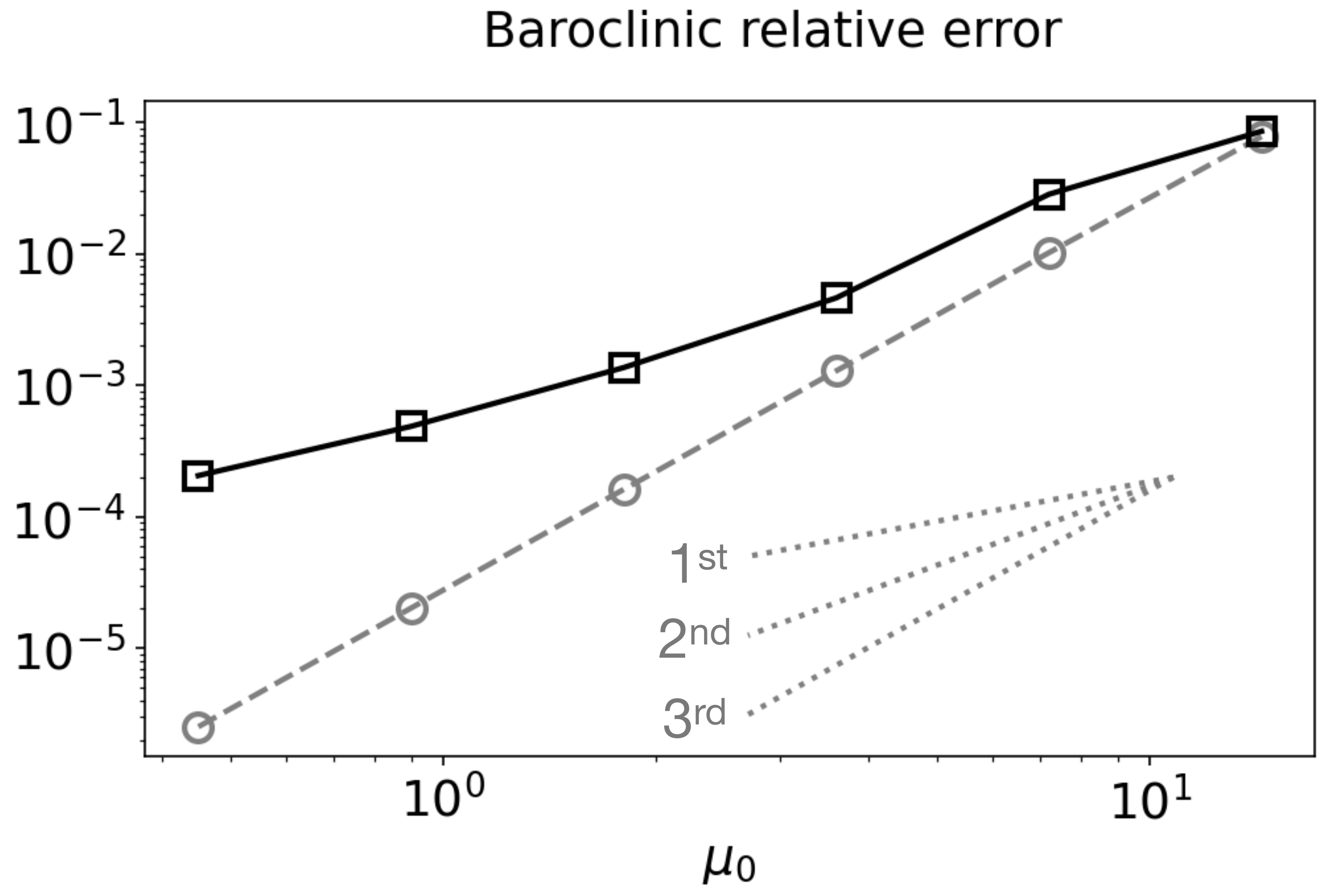
3 - Numerical experiments

With a « toy model »

- The same linearized HPEs that in the analysis, Fourier in x, 2nd-order discretized in z,
- Split-RK3 with First-Stage and Last-Stage strategies implemented using the dissipative FB as 2D scheme,
- The discrete Sturm-Liouville problem is formulated and solved ⟶ exact solution, diag of Split-RK3 solutions.

Split-RK3-FS, with minimum 2D damping



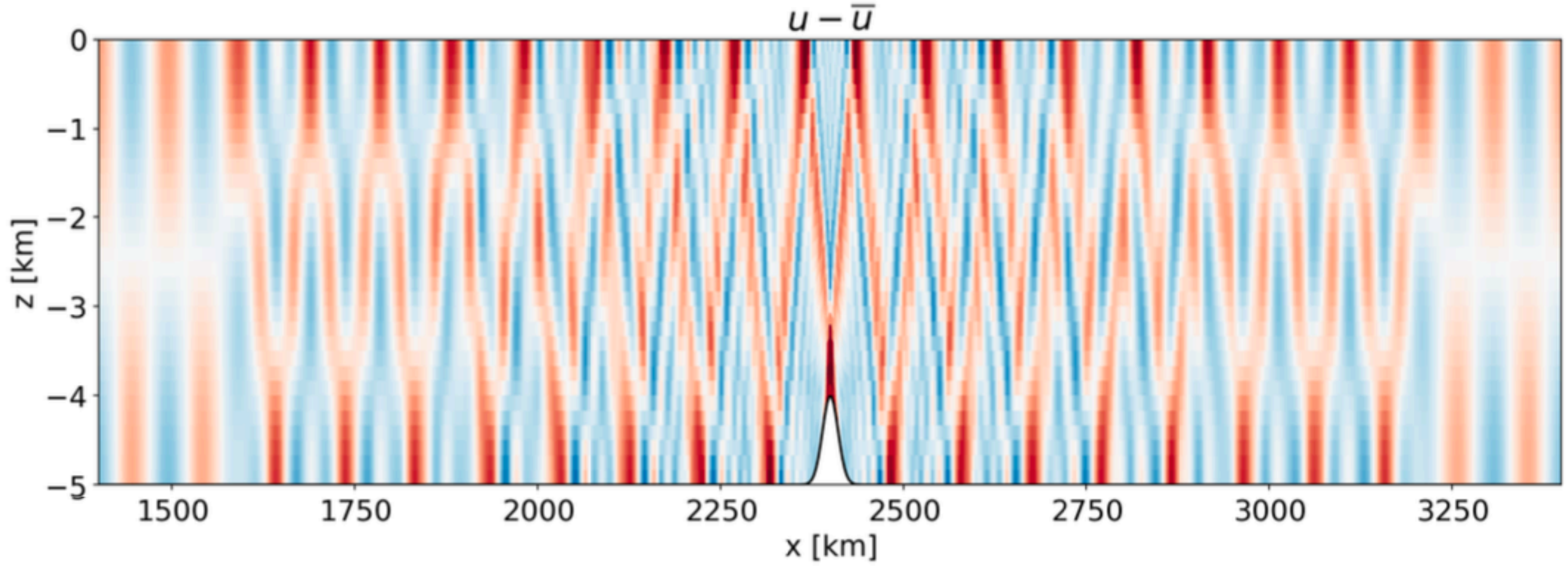


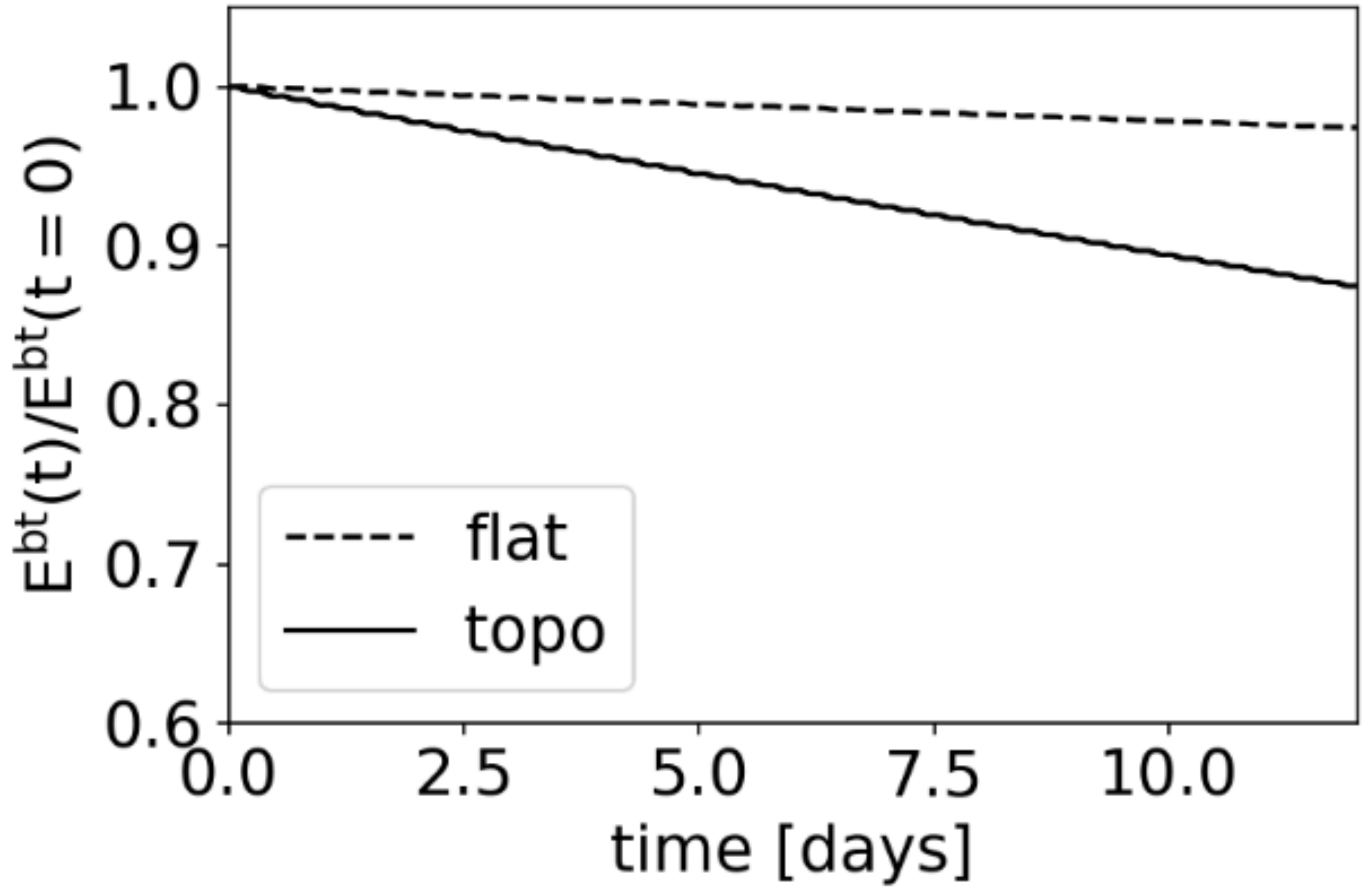
→ Stability and accuracy results in line with previous analysis

3 - Numerical experiments

With an « OGCM prototype »

- The non-linear HPEs, with terrain-following coordinates,
- Split-RK3 with First-Stage strategy implemented using the Generalized FB with tunable, 2nd-order at large scales, dissipation as 2D scheme (Demange et al. 2019),
- An idealized internal tides configuration.





→ Similar results to those obtained with CROCO (Leapfrog-AM3-based split-explicit)

Summary

- Feasibility of building reliable algorithms based on simple Runge—Kutta schemes for the 3D part,
- Several strategies for arranging the 2D/3D coupling with the multi-stage time step,
- Depending on the strategy, different types of instabilities, more or less efficiently controlled by 2D damping,
- The First-Stage and Last-Stage strategies, with corrections at intermediate stages, are promising,
- The results of numerical experiments in agreement with the results of the analysis.

- see S. Téchené’s presentation : implementation of Split-RK3 with the First-Stage strategy in NEMO

- These results and more in a paper to be submitted
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