Some insights on the spurious numerical mixing of the time-stepping of advection schemes

Adrien Garinet

Patrick Marsaleix, Marine Hermann

COMMODORE Workshop 2024

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Outline

Advection schemes

Spurious numerical mixing → Semi-discrete framework → Space-time framework

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Spurious numerical mixing

Semi-discrete framework

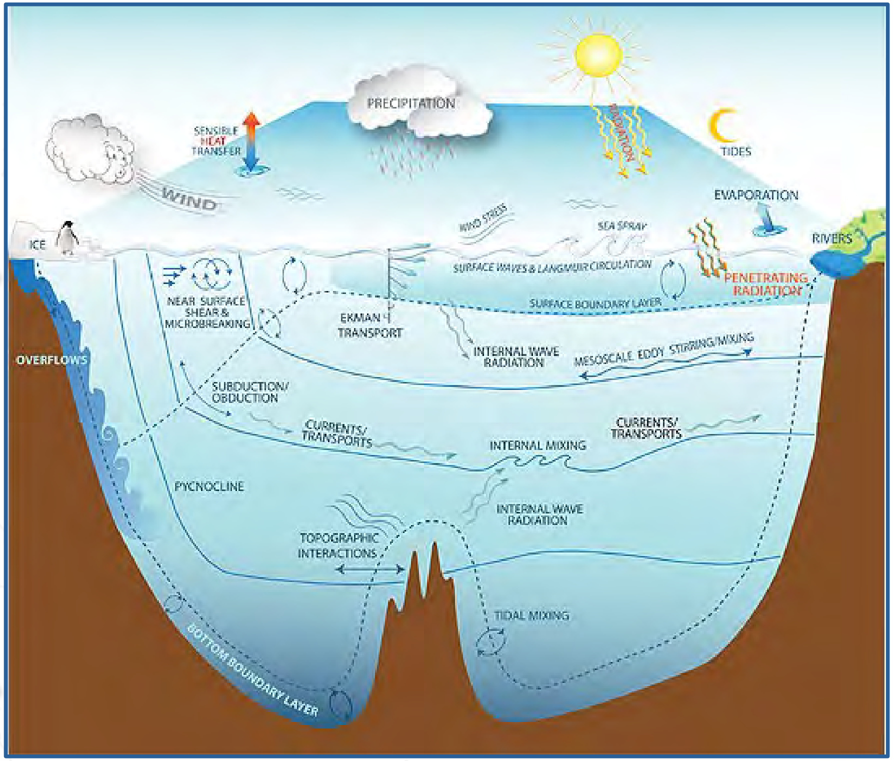
Space-time framework

Mixing in the ocean

In numerical models :

Equations

Numerical methods



Source : GFDL / Ocean Mixing

Mixing in the ocean

In numerical models :

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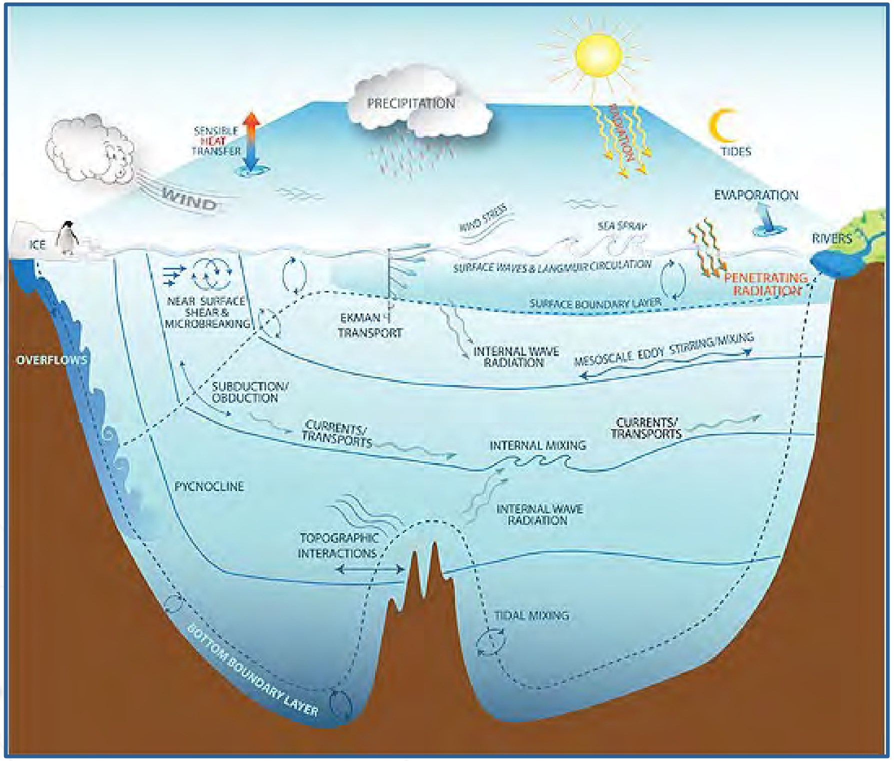
Numerical methods

Errors !

Numerical mixing



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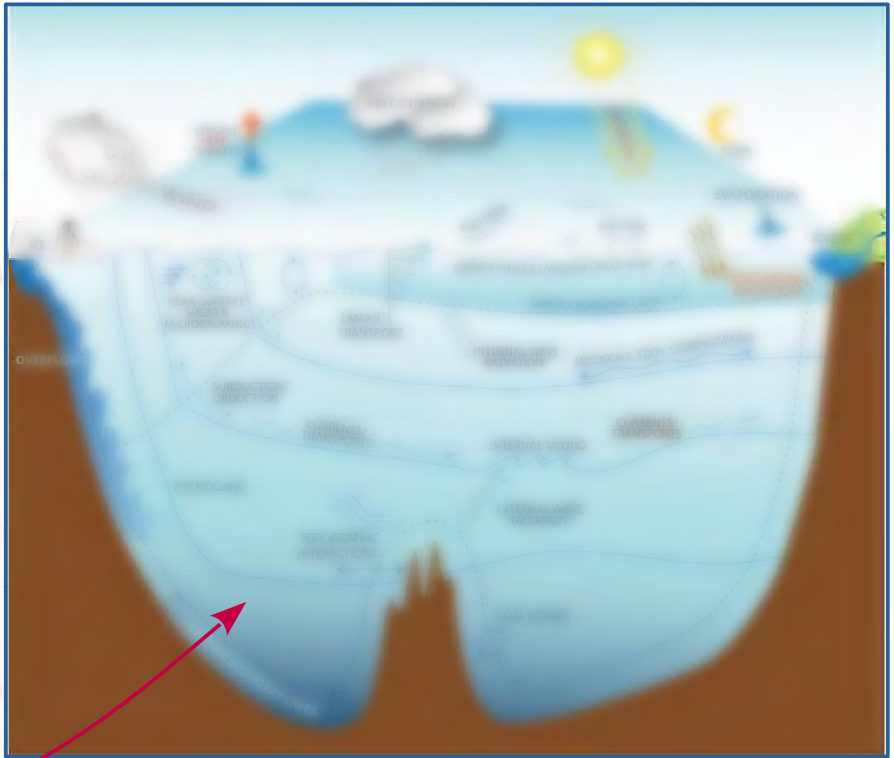

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Spurious numerical mixing

- Well-known problem in “fixed coordinates models

  [Griffies et al. (2000)]

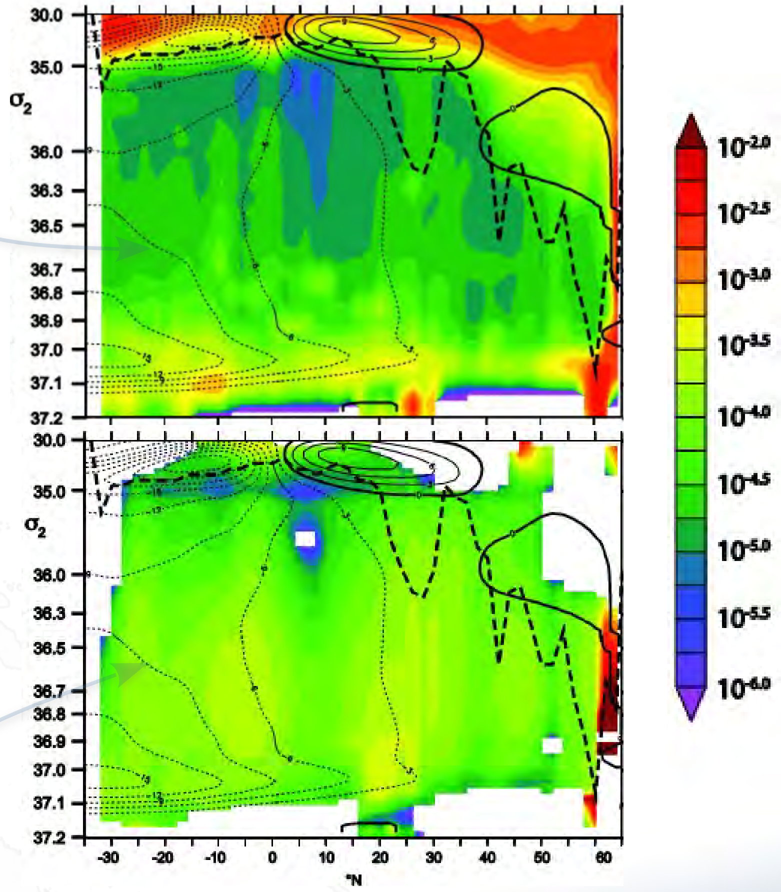
- Quantification in academical simulations

  [Burchard & Rennau (2008)]
  [Gibson et al. (2017)]
  [...]

- Quantification in global models

  [Lee et al. (2002)]
  [Megan (2018, 2023)]
  [Holmes et al. (2021)]
  [...]

K_{phy}



K_{eff}

Source : Megan (2018)

Well-known problem in "fixed coordinates models

[Griffies

Quantification in academical simulations

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Global models

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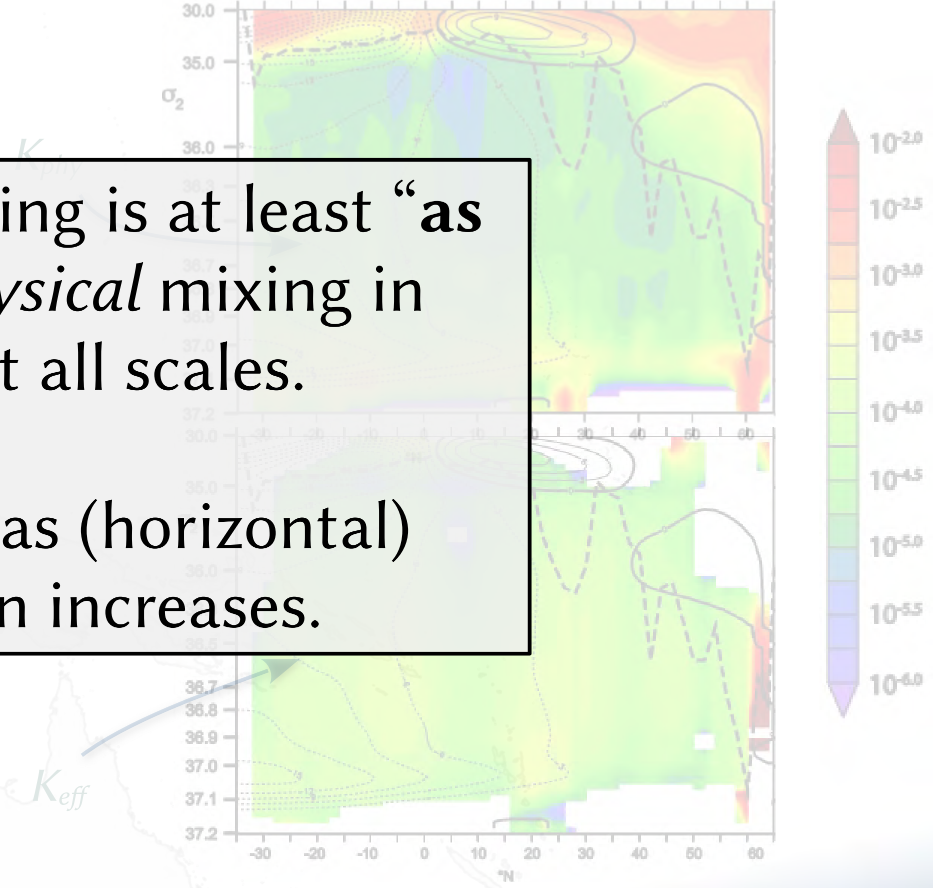
Numerical mixing is at least "**as high as**" *physical* mixing in models at all scales.

It increases as (horizontal) resolution increases.



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Spurious numerical mixing

Usual suspect 🚓 (in fixed coordinate models) :

$$\boxed{\partial_t s = -\partial_z (w s)} + \partial_z [K_z \partial_z s] + \ldots$$

Vertical advection of tracer

Spurious numerical mixing

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Vertical advection of tracer

Once discretized :

$$\mathrm{Diff}_{\mathrm{num}} \propto w$$

vertical speed

Spurious numerical mixing

Usual suspect (in fixed coordinate models) :

A lot of work has been carried out to improve **vertical coordinates**.

Now, can we also work on **advection schemes** ?

Once discretized :

$$\mathrm{Diff}_{\mathrm{num}} \propto w$$

vertical speed

Outline

Spurious numerical mixing → **Semi-discrete framework** → Space-time framework

Semi-discrete advection equation

- Finite volume formulation

$$\frac{d}{dt}[s_j] = \mathrm{ADV}[w, s]_j$$

with, for linear schemes

$$\mathrm{ADV}[w, s]_j = \underbrace{\mathcal{A}[w, s]_j}_{\text{transport}} + \underbrace{\mathcal{D}[w, s]_j}_{\text{damping}}$$

controls dispersion errors

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e.g. 3rd order upwind-biased scheme

$$\mathrm{ADV}_{\mathrm{UP3}} = \mathrm{C}_4 + \mathrm{D}_4$$

$$\sim \frac{\partial^4 s}{\partial z^4}$$

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$$\mathrm{ADV}_{\mathrm{UP3}} = \mathrm{C}_4 + \mathrm{D}_4$$

$$\sim \frac{\partial^4 s}{\partial z^4}$$

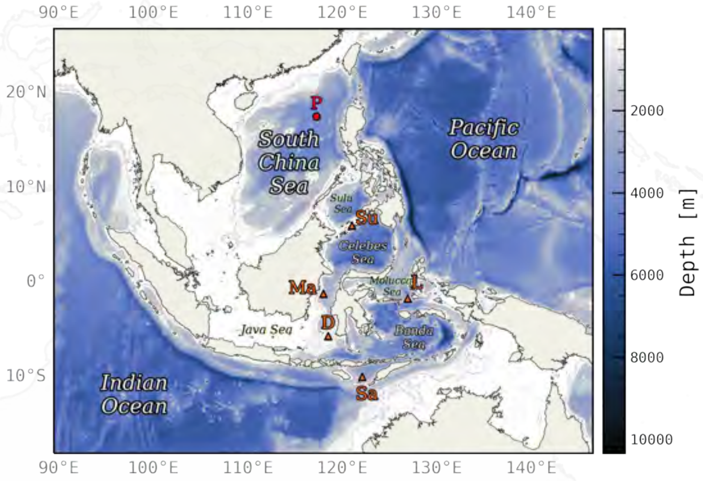
not scale selective enough !

Some results

- Using

$$\mathcal{D}[w, s]_j \sim \mathrm{D}_6$$

$\longrightarrow$ good results in a **5 km** resolution model of the **South-East Asian Seas** (strong internal tides)



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Hereafter, we refer to the scheme as UP3-F ; implemented in the **Symphonie** ocean model.

More details can be found in :

Garinet *et al.* (2024) Spurious numerical mixing under strong tidal forcing: a case study in the South East Asian Seas using the Symphonie model (v3.1.2)

Validation in a realistic simulation

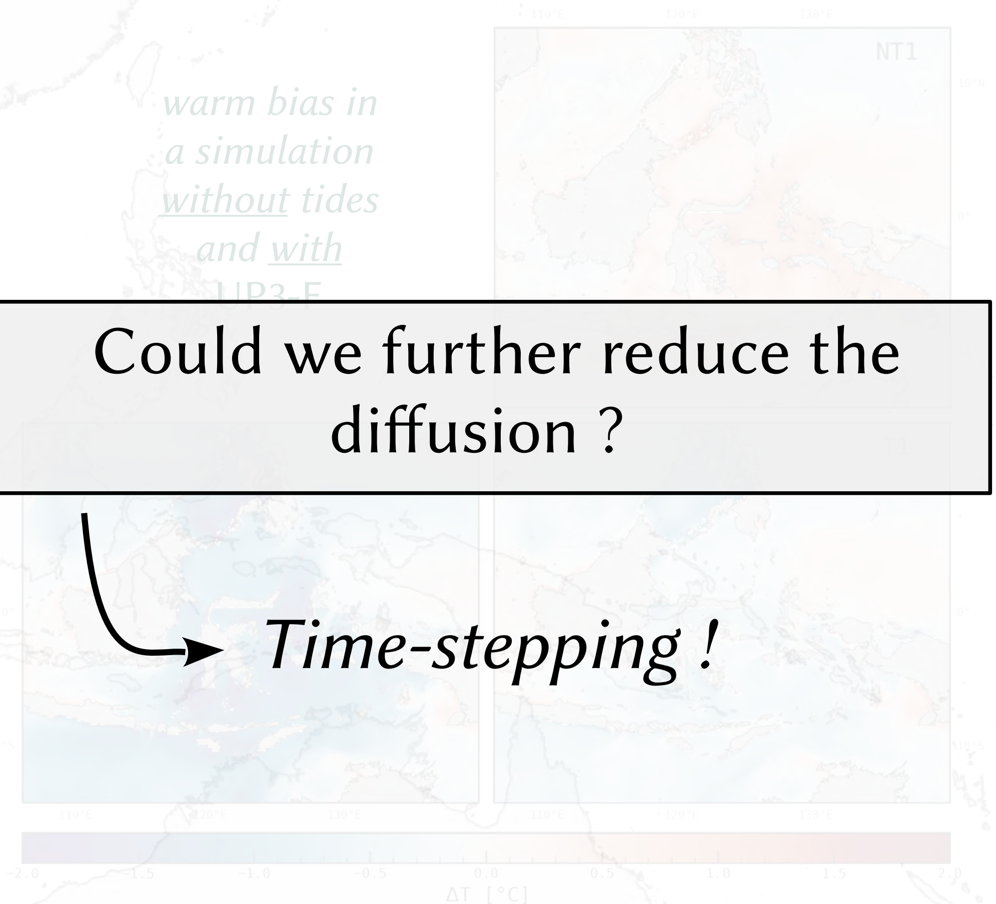
warm bias in a simulation <u>without</u> tides and <u>with</u> UP3-F

Could we further reduce the diffusion ?

Time-stepping !

cold bias in a simulation <u>with</u> tides and <u>without</u> UP3-F

<u>with</u> tides and <u>with</u> UP3-F : ~ ok



Mean SST bias over 2017-2018 between simulations and OSTIA product.
Negative bias indicates underestimation by the model.

Outline

Spurious numerical mixing → Semi-discrete framework → **Space-time framework**

Analysis in a coupled space-time framework

- Formally, a time-stepping scheme writes

$$s^{n+1} = \mathcal{F}(\{s^m\}_{m \leq n}, \mathrm{ADV})$$

We look for solutions in the form

$$\rho^n e^{i\theta}$$

Amplification factor

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Angular wavenumber : $\theta = 2\pi / \lambda_N$

Wavelength in number of grid points

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Amplification factor

Angular wavenumber : $\theta = 2\pi / \lambda_N$

Wavelength in number of grid points

We are interested in

$$\delta\rho = 1 - |\rho|$$

Ideally $\delta\rho(\theta \sim 0) \approx 0$ while $\delta\rho(\theta \sim \pi)$ should be "large enough".

(*i.e.* $\leq 10^{-4}$)

Analysis in a coupled space-time framework

- Noting

$$\Delta t \mathrm{A}\hat{\mathrm{D}}\mathrm{V}(\theta) = \underbrace{\boxed{\mu_r(\theta)}}_{\text{damping}} + i\underbrace{\boxed{\mu_i(\theta)}}_{\text{transport}}$$

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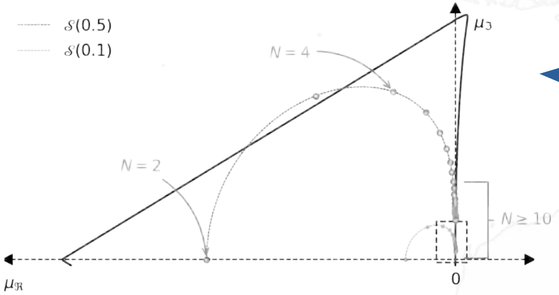
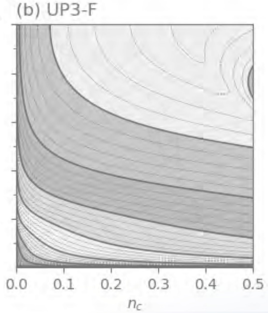
We end up solving for ρ one of these :

$$\begin{cases} \mathcal{T}(\rho, n_c, \theta) & = 0 \\ \mathcal{T}(\rho, \mu) & = 0 \end{cases}$$

Courant number : $n_c = \dfrac{w\Delta t}{\Delta z}$





- In Symphonie : Leapfrog with Robert-Asselin filtering (LFRA)

$$\begin{cases} s^{n+1,*} & = s^{n-1} - 2\frac{\Delta t}{\Delta x}\left[\mathcal{A}[s^{n,*}] + \mathcal{D}[s^{n-1}]\right] \\ s^{n} & = \chi s^{n+1,*} + (1-2\chi)s^{n,*} + \chi s^{n-1} \end{cases}$$

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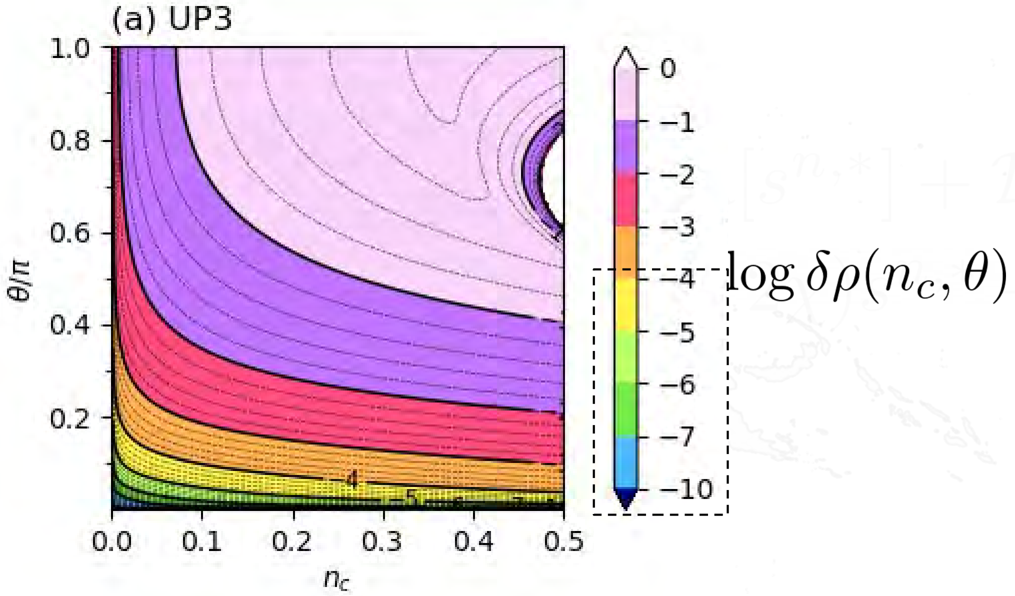

Amplification factor error as a function of Courant number and angular wavenumber for UP3 advection scheme, used along with LFRA time-stepping (χ = 0.05).

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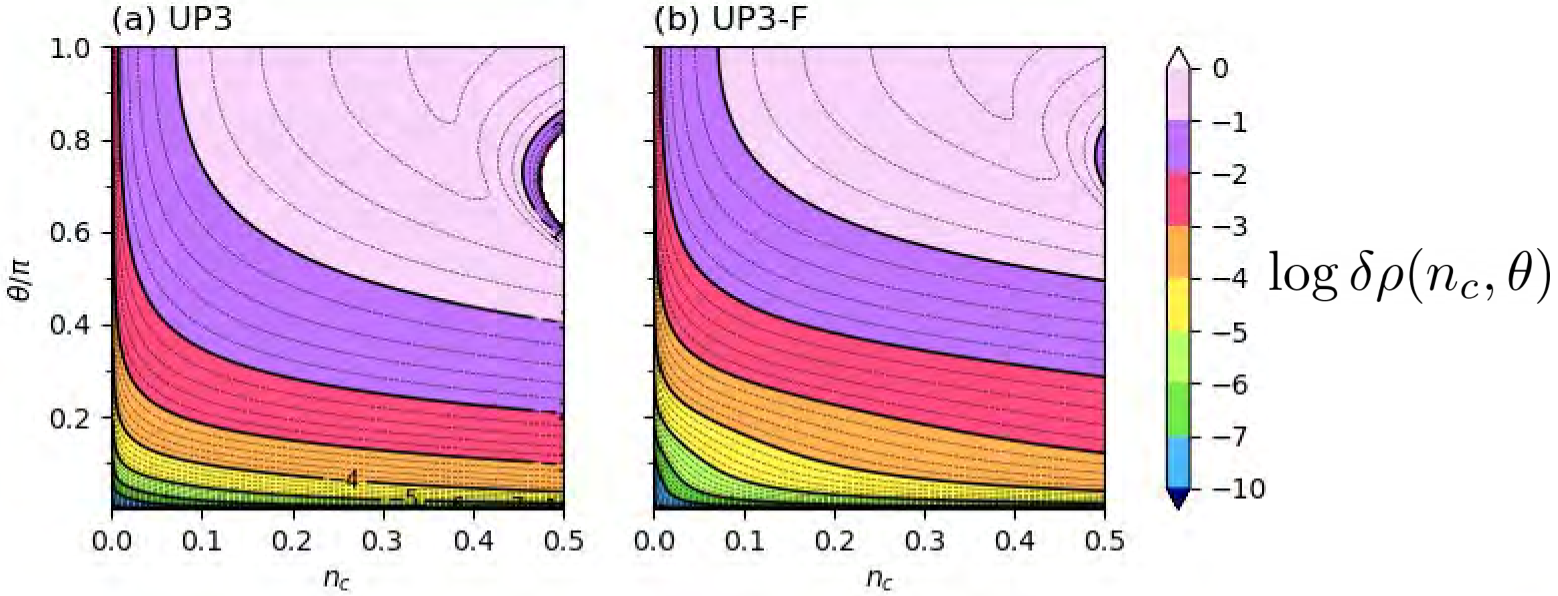
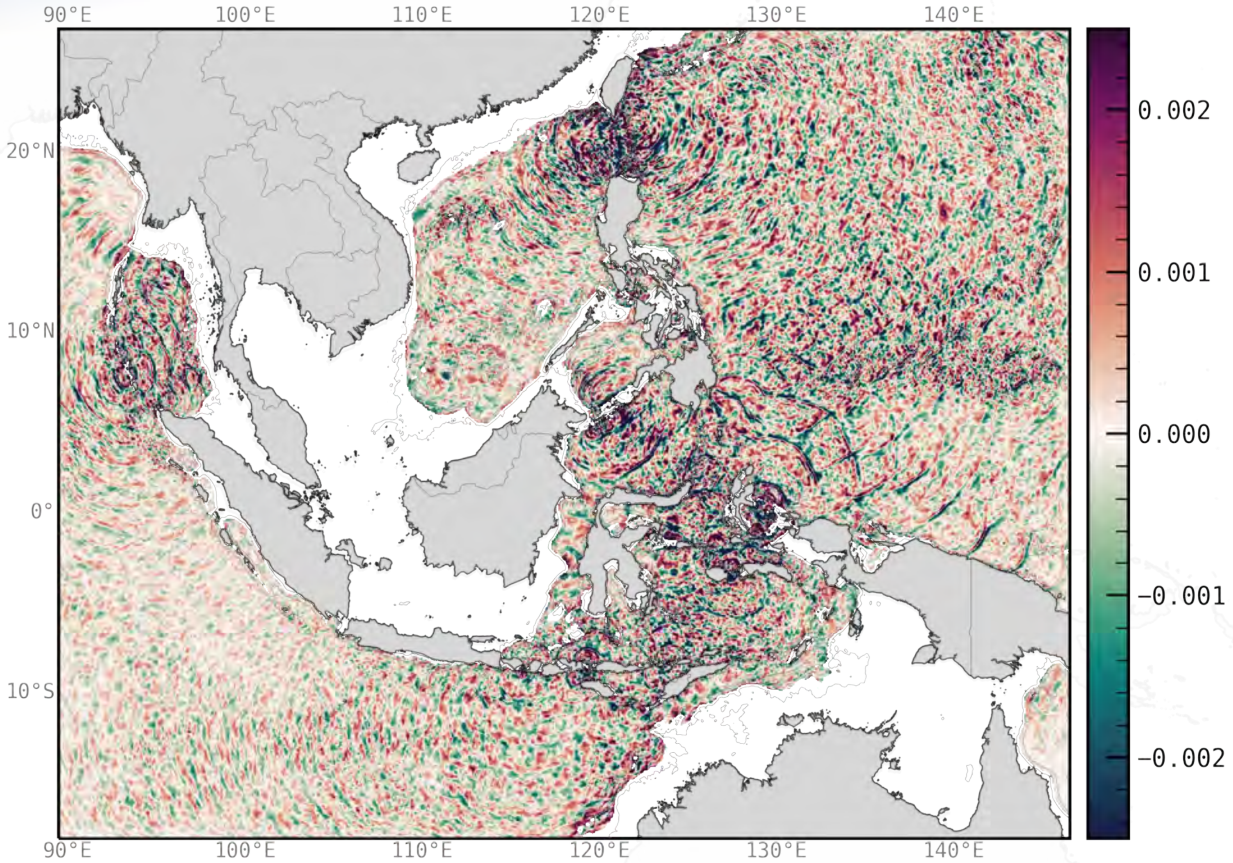


Amplification factor error as a function of Courant number and angular wavenumber for both UP3 and UP3-F advection schemes used along with LFRA time-stepping (χ = 0.05).



Instant vertical velocities at 200 m depth in Symphonie SEA simulation [m.s^{-1}]

Max admissible time-step :

$$\Delta t \approx 630 \text{ s}$$

In the ocean interior, where

$$\Delta z \approx 10 \text{ m}$$

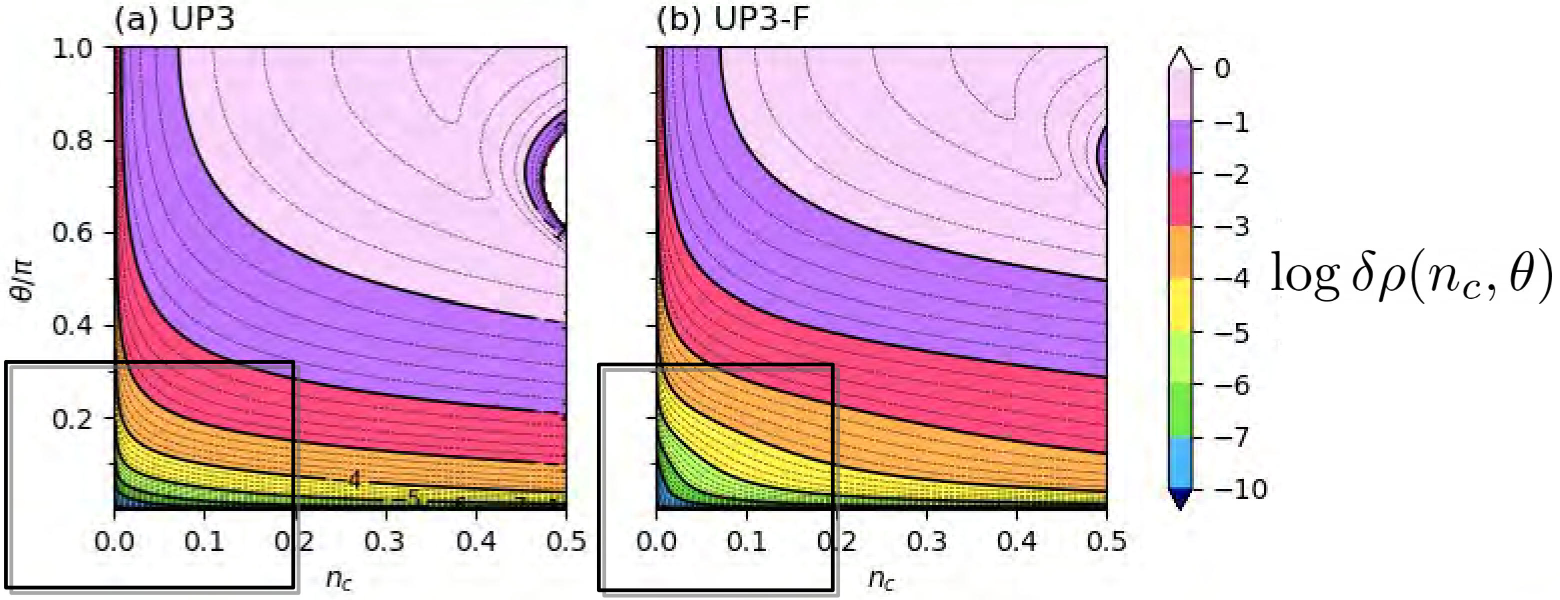
and assuming

$$w_{\max} \approx 0.003 \text{ m.s}^{-1}$$

we thus have

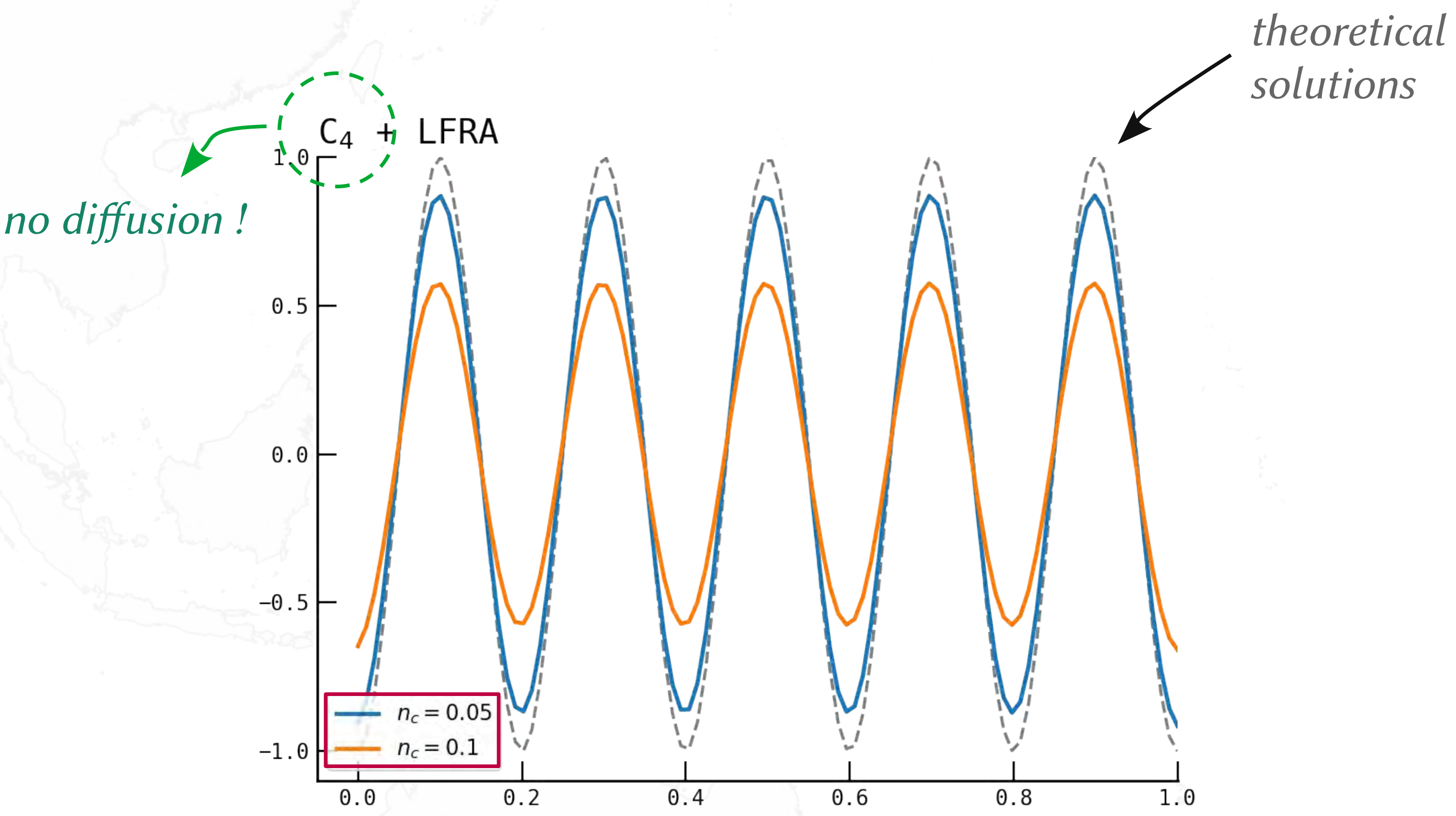
$$\boxed{n_c \leq 0.2}$$

- In Symphonie : Leapfrog with Robert-Asselin filtering (LFRA)



Amplification factor error as a function of Courant number and angular wavenumber for both UP3 and UP3-F advection schemes used along with LFRA time-stepping.

- Still :

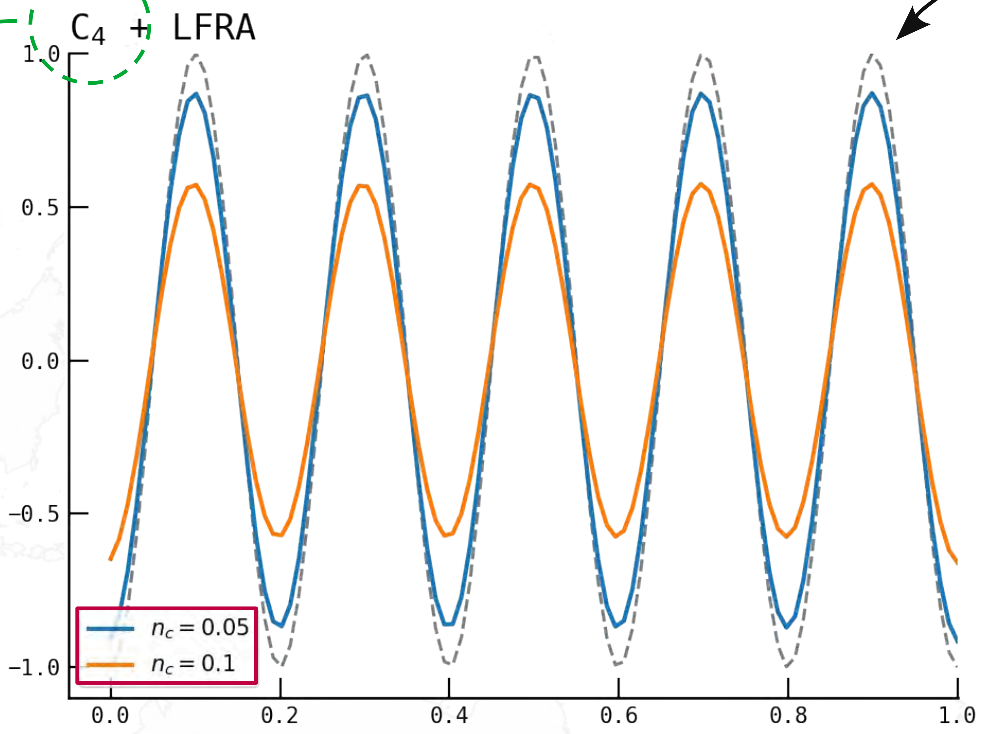


Simulation of 60 back and forth motions of a 20 grid points sine profile.

- Still :

no diffusion !

theoretical solutions



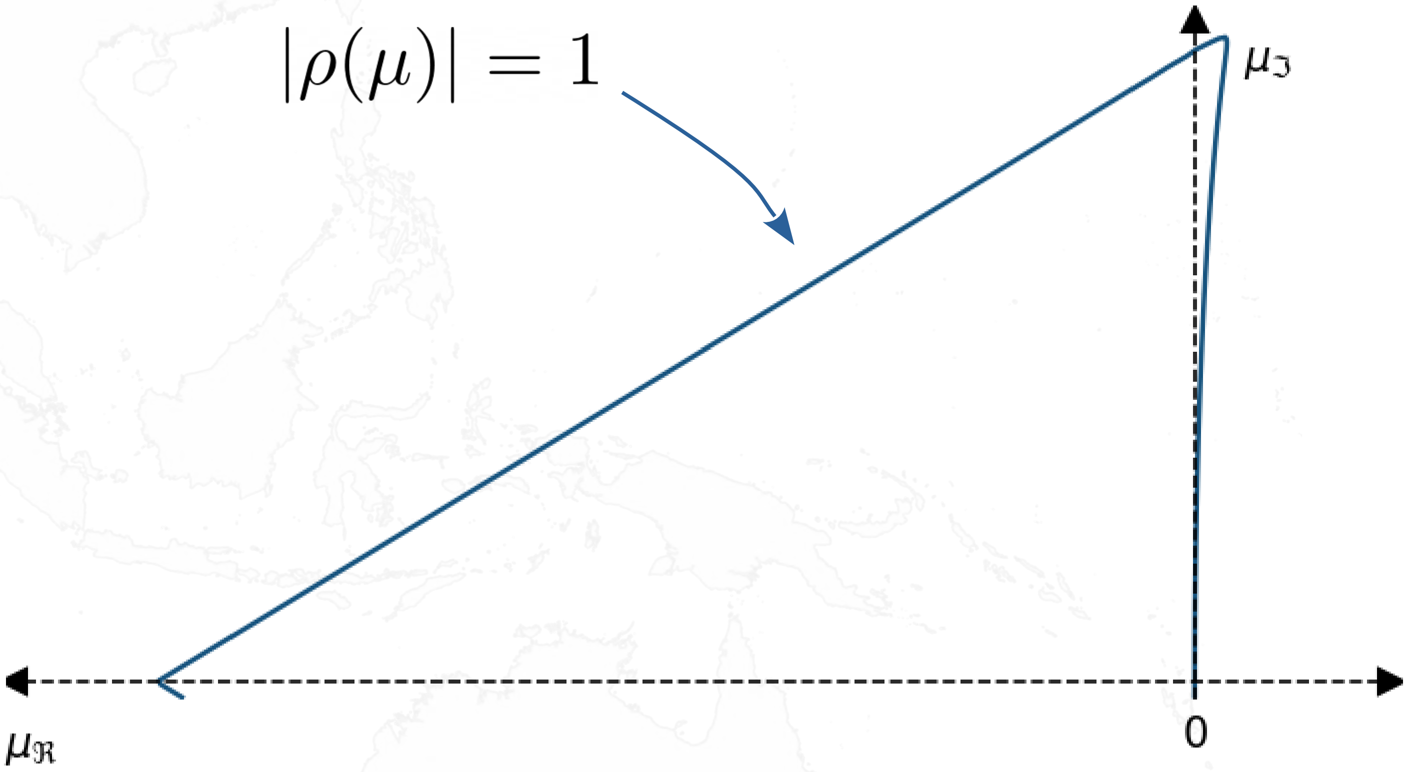
→ *higher Courant number*

=

more diffusion

Simulation of 60 back and forth motions of a 20 grid points sine profile.

Stability domain



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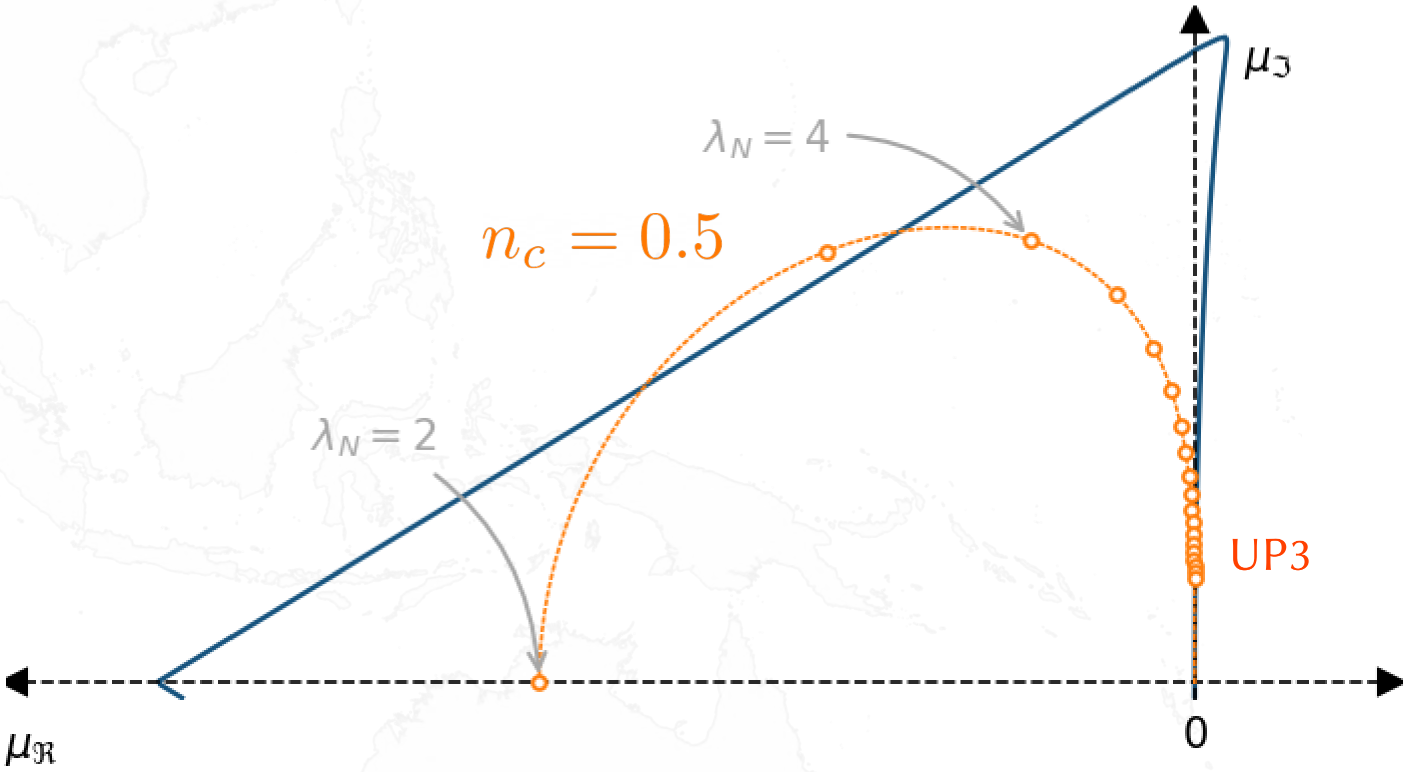

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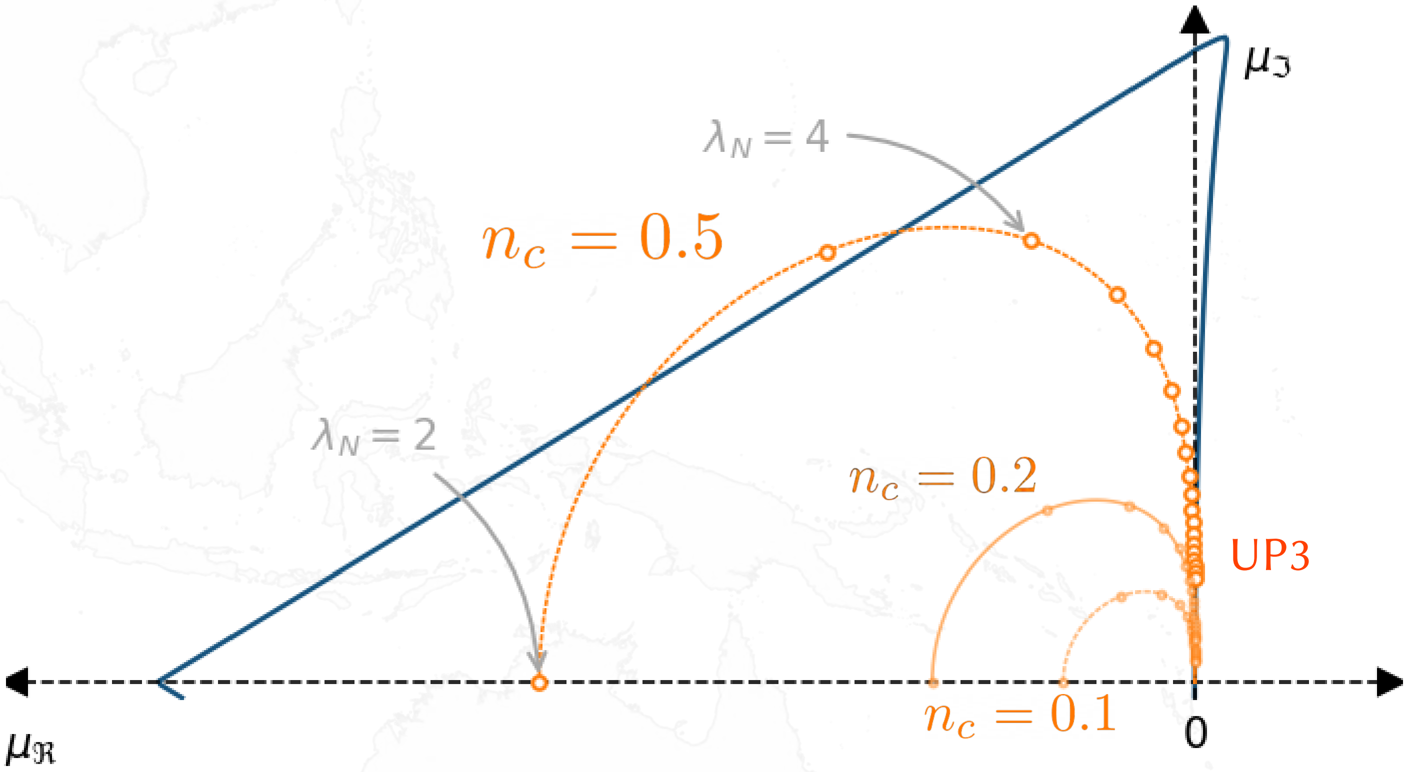

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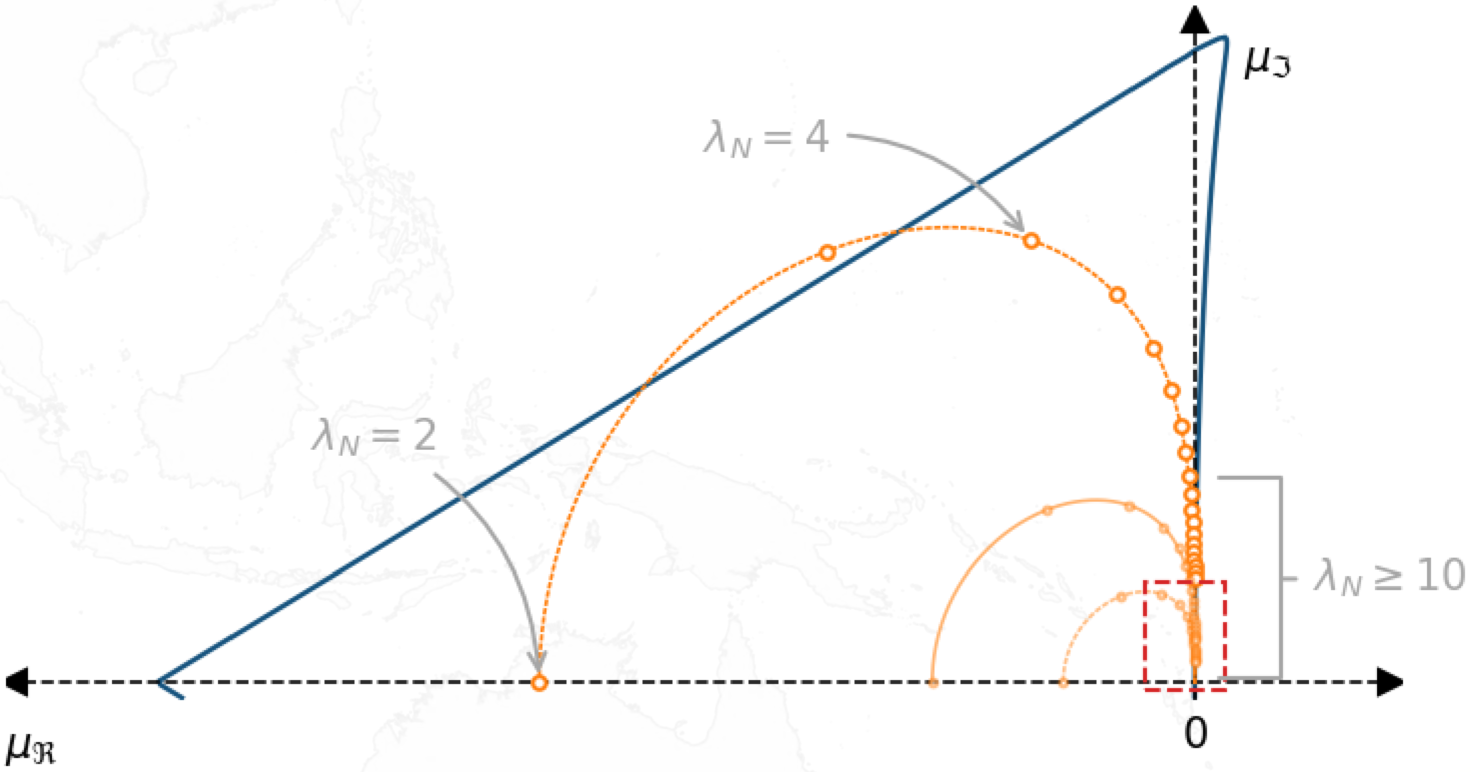

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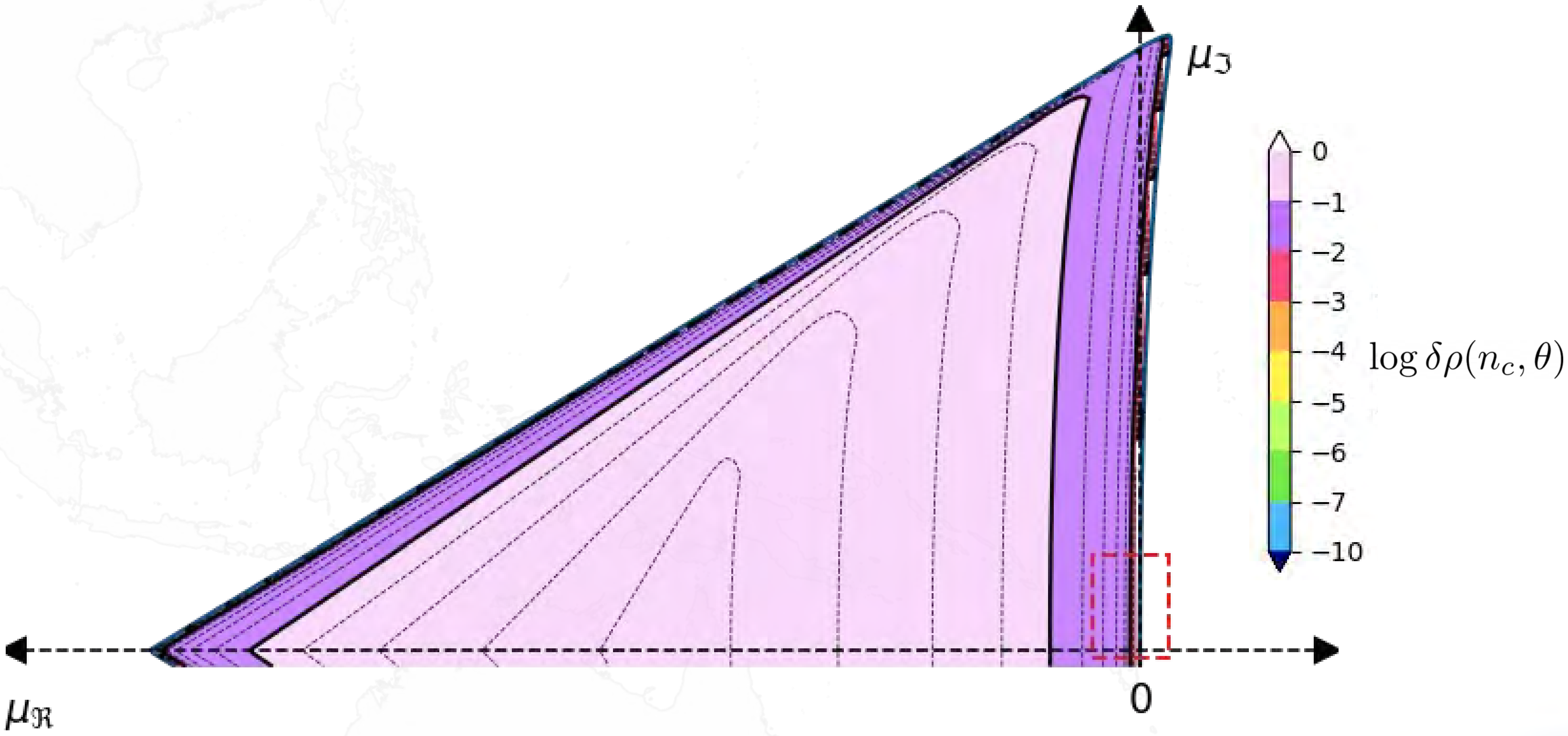
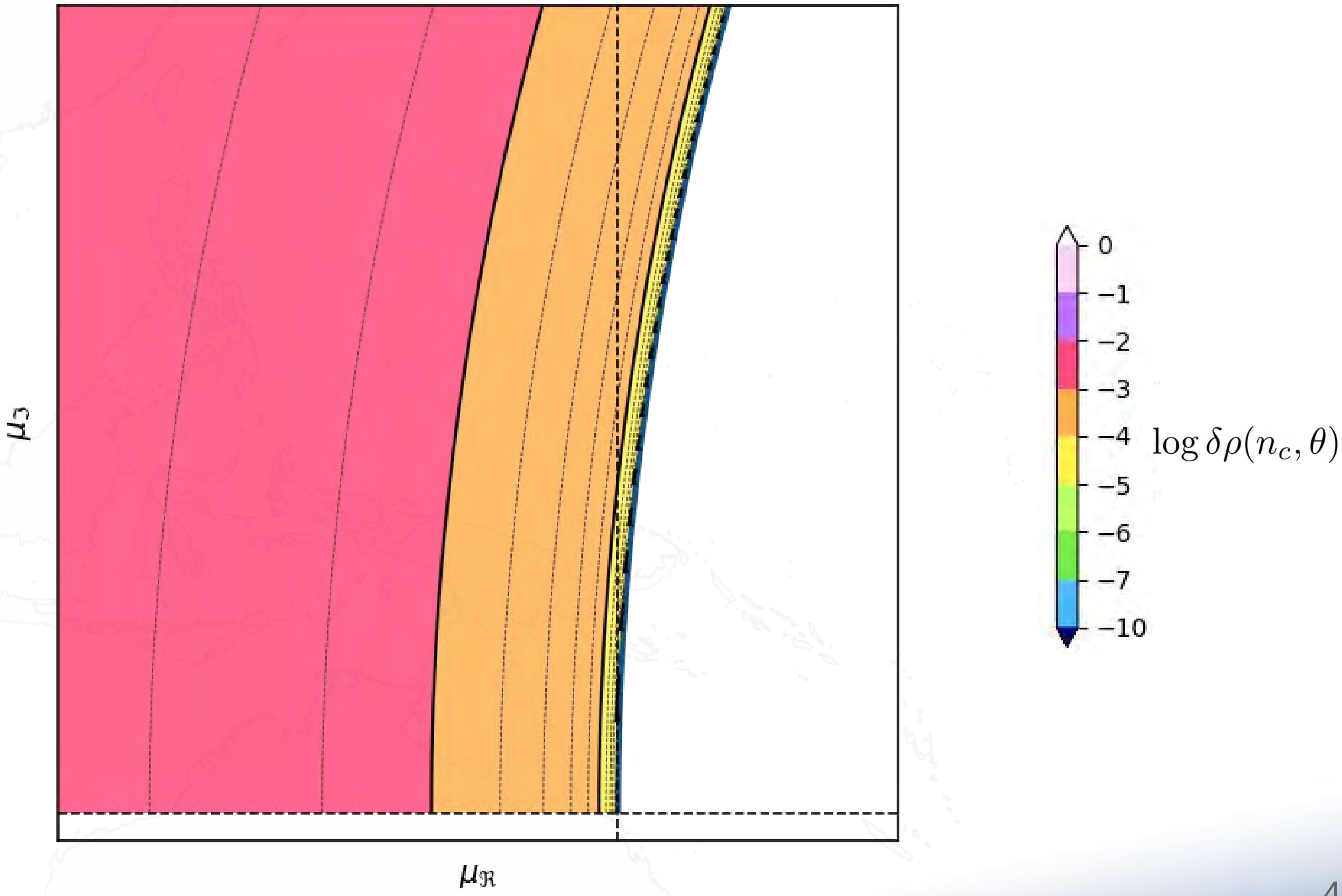
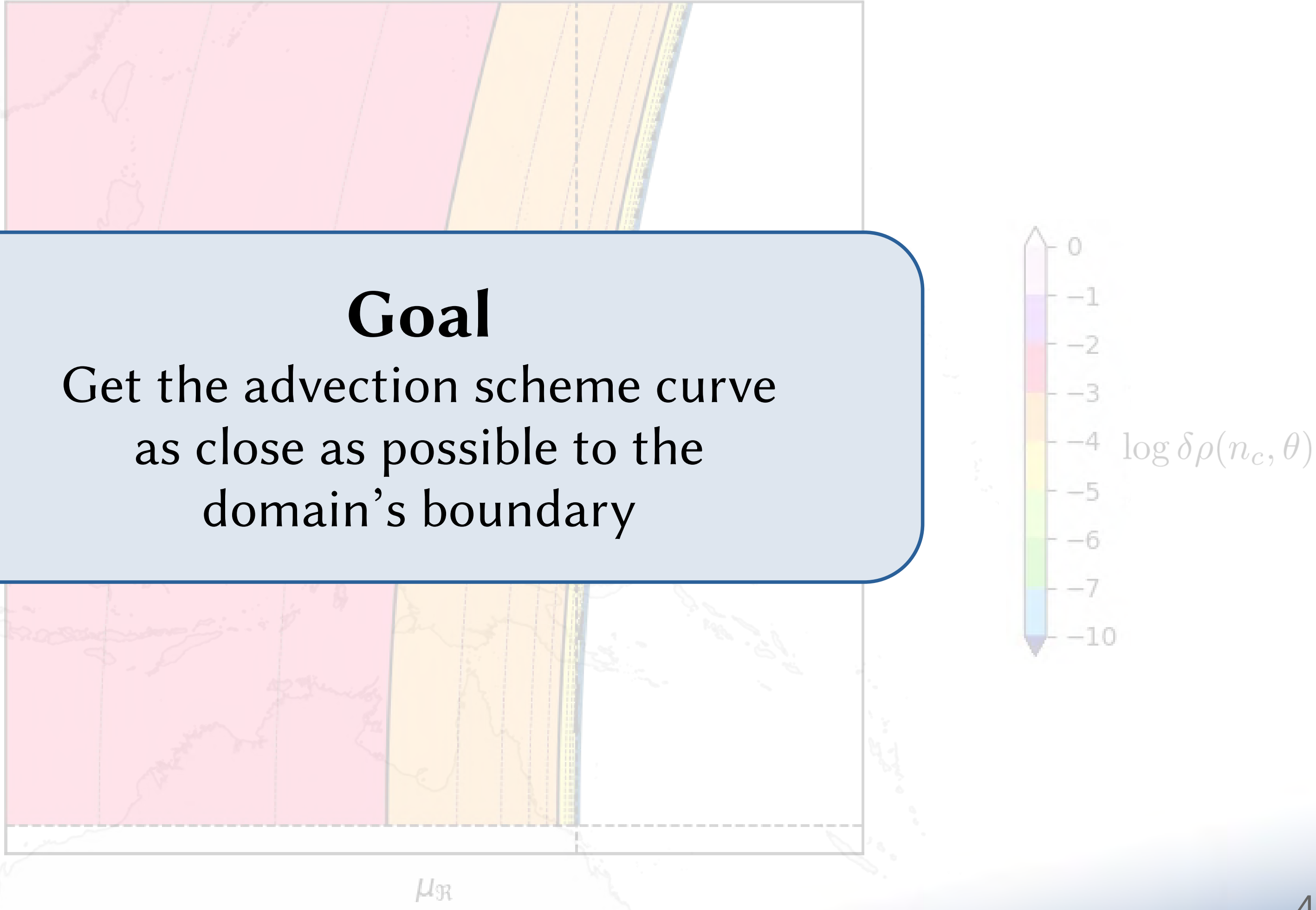
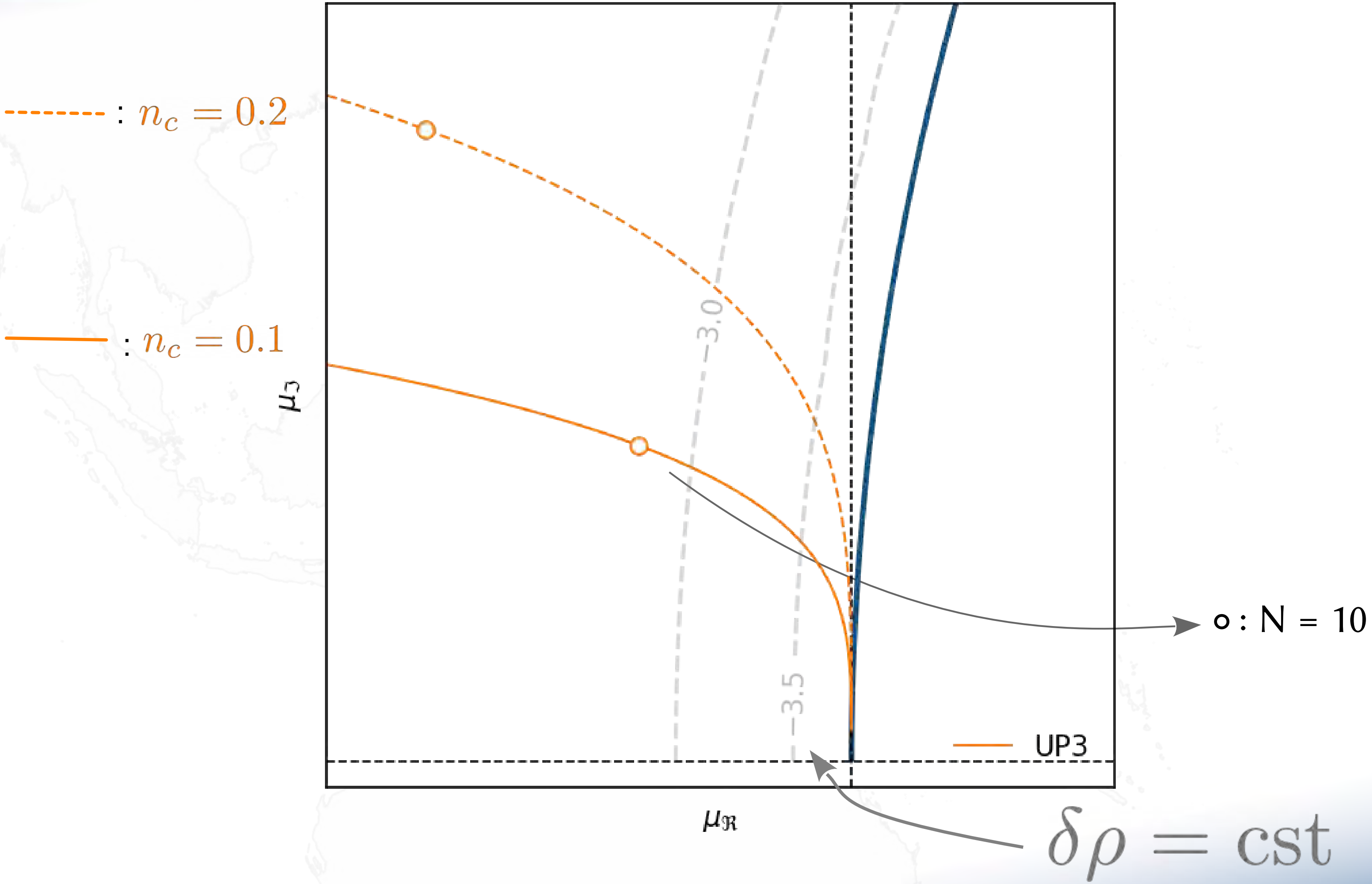



$\mu_{\Im}$
$\mu_{\Re}$
0
−1
−2
−3
−4
−5
−6
−7
−10
$\log \delta\rho(n_c, \theta)$

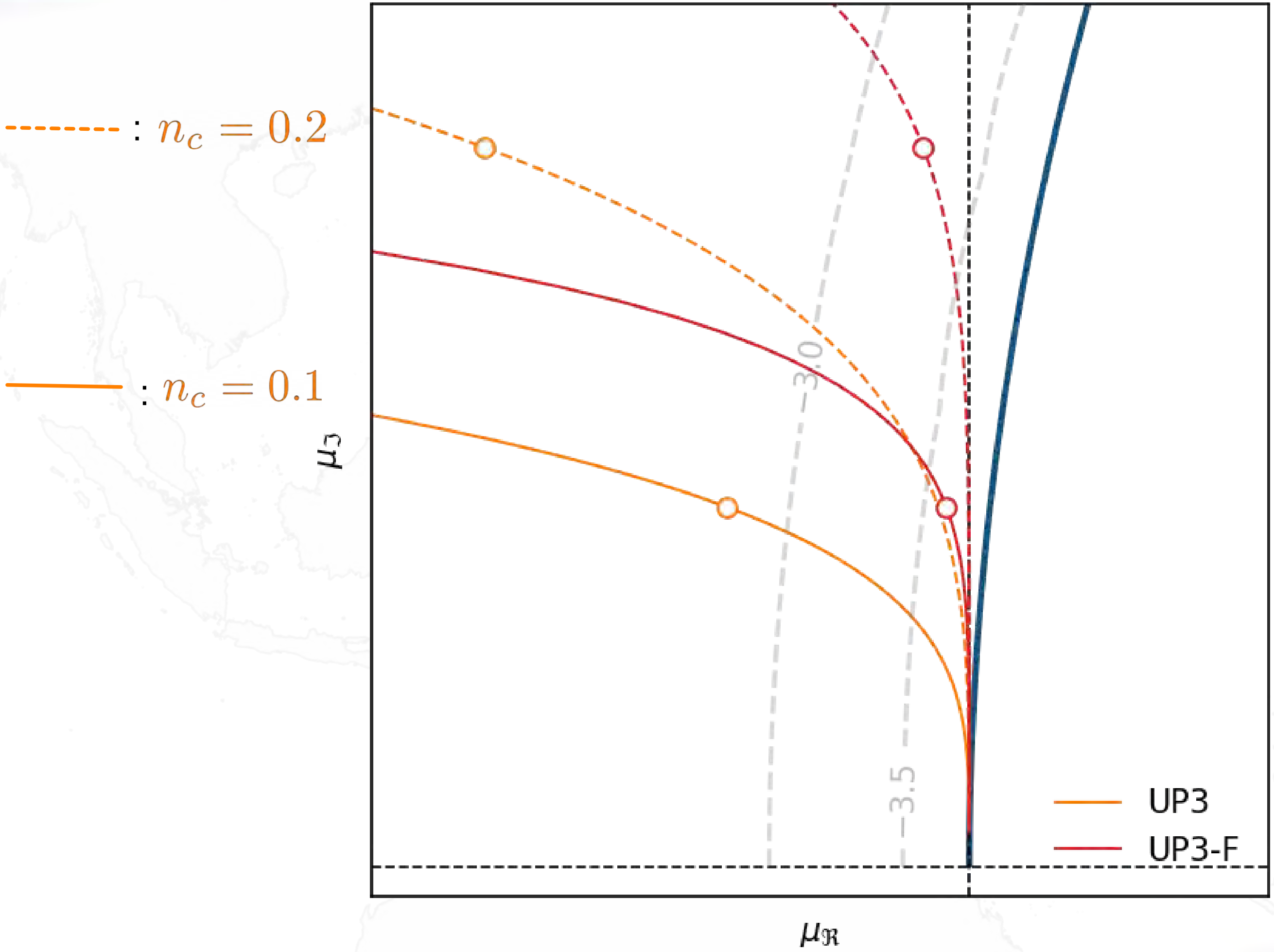


Goal

Get the advection scheme curve as close as possible to the domain's boundary



: $n_c = 0.2$
: $n_c = 0.1$
$\mu_{\Im}$
$\mu_{\Re}$
-3.0
-3.5
UP3
o : N = 10
$\delta\rho = \mathrm{cst}$


: $n_c = 0.2$
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$\mu_\Im$
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UP3
UP3-F

$\text{- - - - -} : n_c = 0.2$

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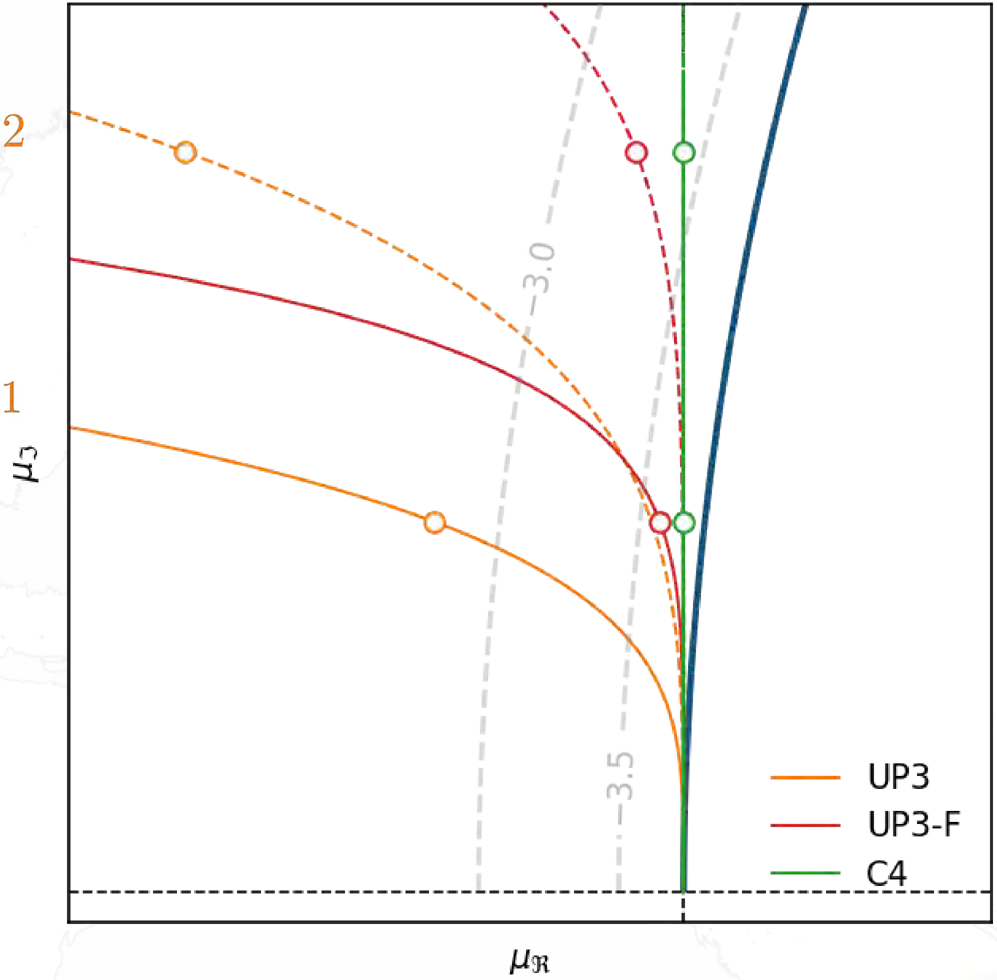

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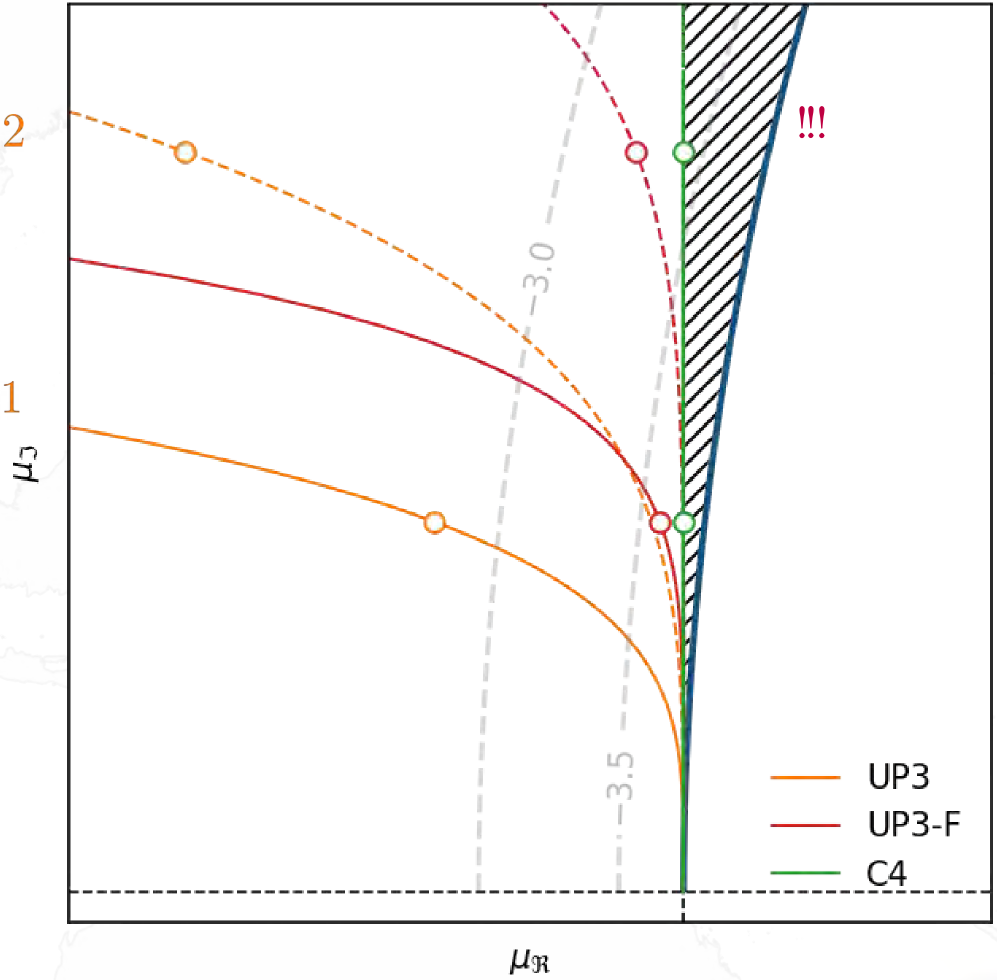

Backscattering formulation

We want to introduce *anti-diffusion* at certain scales, i.e. have

$$\mu_r(n_c, \theta) > 0$$

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We formulate the diffusive part as such

$$\mathcal{D}[s]_j = \mathrm{D}_4 \cdot (\mathbf{1} - \overset{>1}{\alpha}\phi)[s]_j$$

where

$$\left.\phi[s]_j = \frac{1}{4}(s_{j+1} + 2s_j + s_{j-1})\right\}\text{low-pass filter}$$

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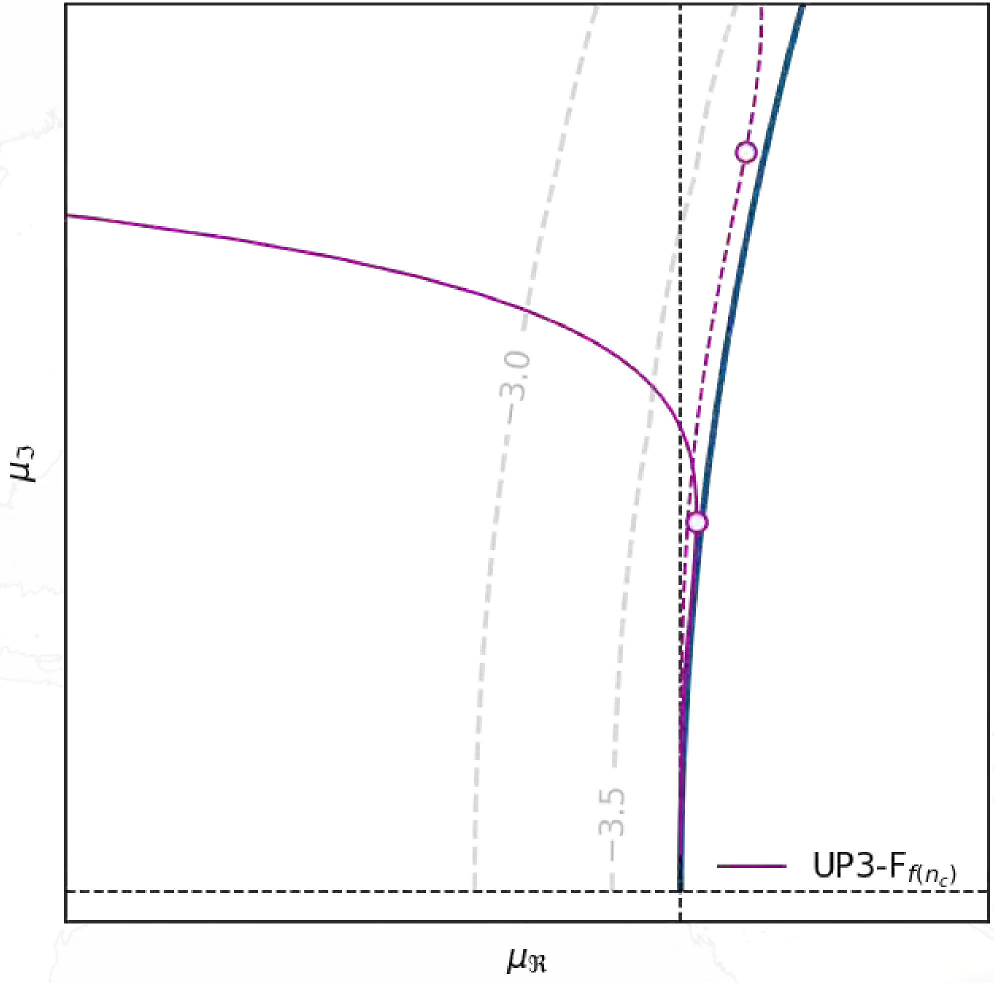
Thus : $$\boxed{\alpha(n_c) = 1 + n_c^{\upsilon} e^{-\xi n_c}}$$ *to be determined !*



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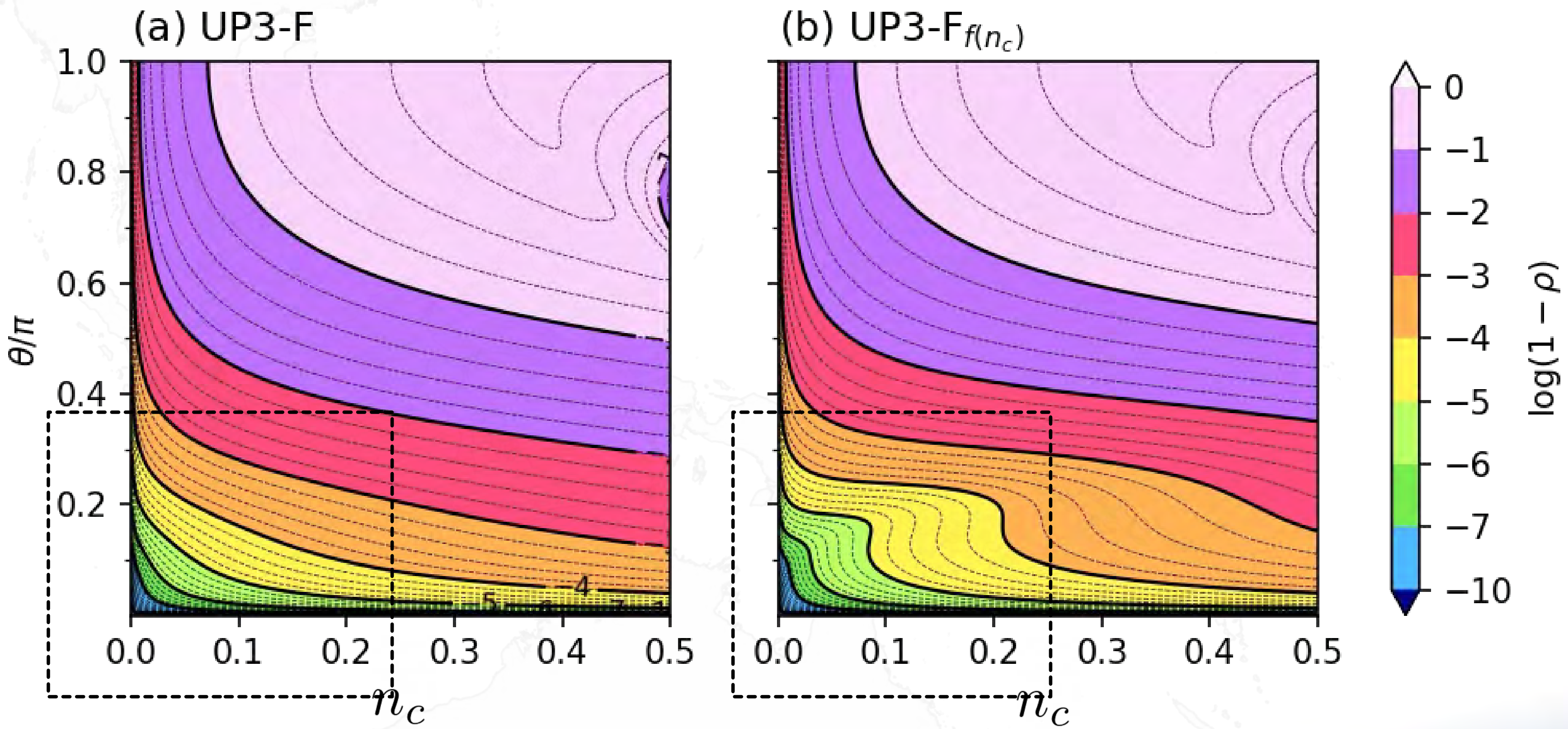
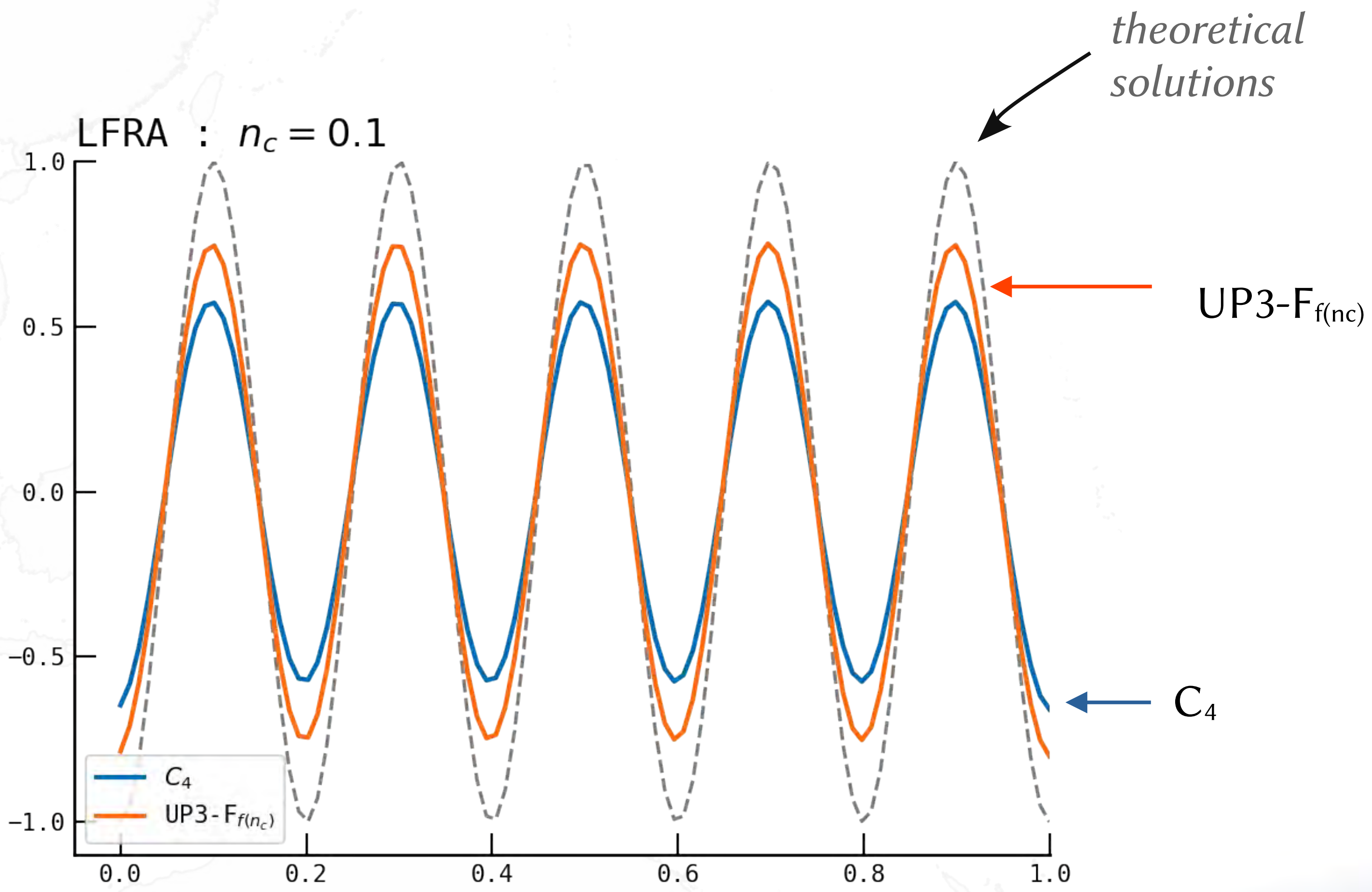
−3.5
UP3-F$_{f(n_c)}$
$\mu_{\Im}$
$\mu_{\Re}$

Backscattering formulation



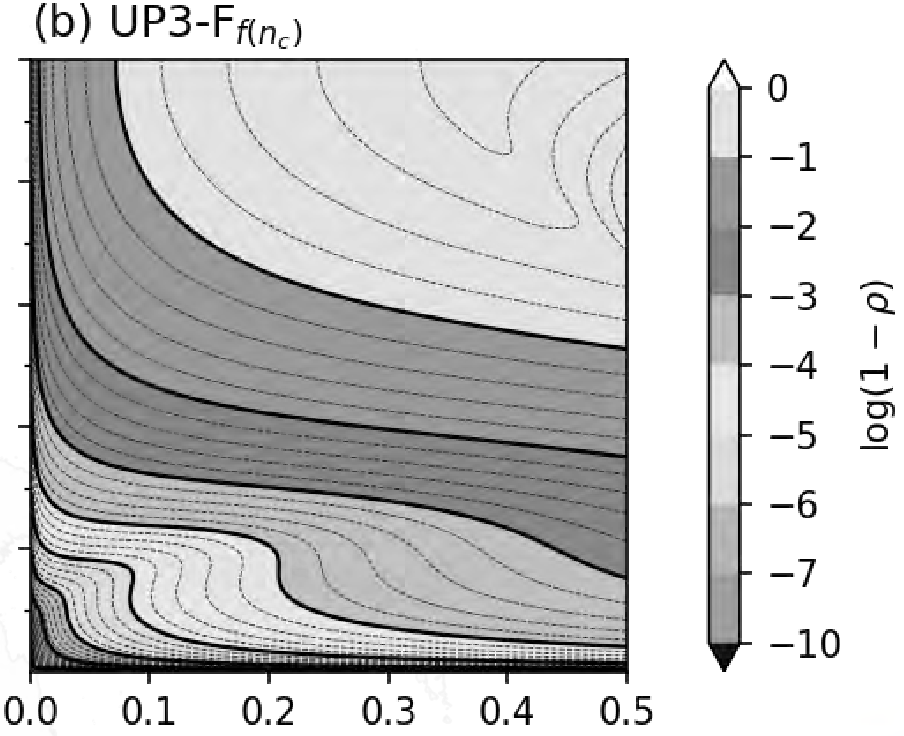
- In 1D tests :



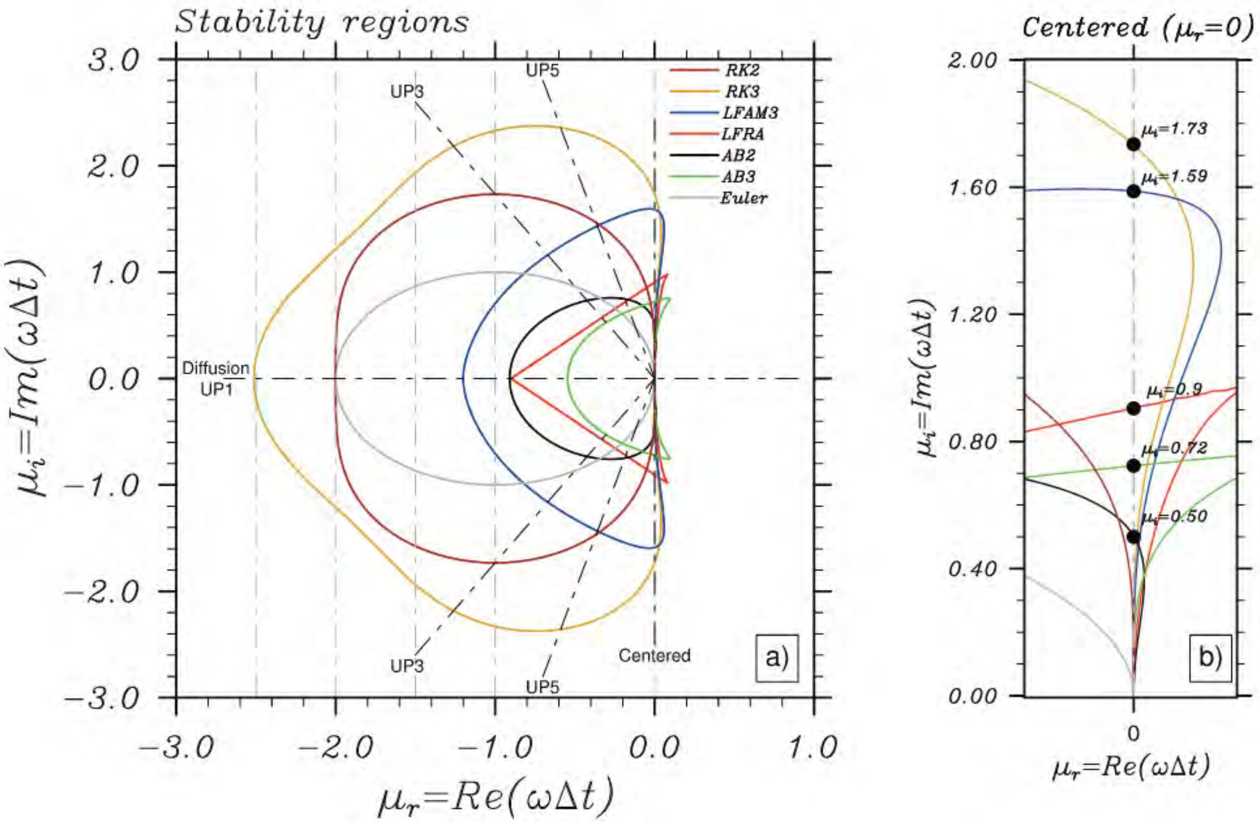
→ Still some work to do before full integration in Symphonie

Conclusions

- New way of formulating the spatial component of advection schemes to make up for the diffusion of the time-stepping
  - *anti-diffusion* can be added;
  - the diffusive component does **not** have to depend **linearly on the speed**.



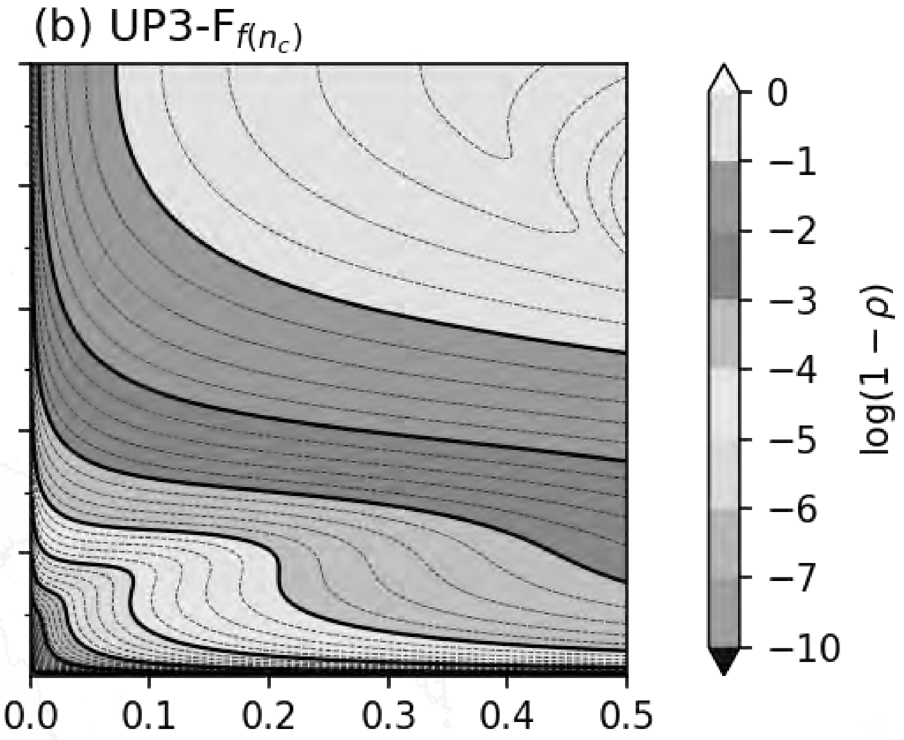
Conclusions



Source : Lemarié *et al.* (2015)

Some more thoughts ?

- *Numerical* mixing seems to be "as high as" *physical* mixing :
  - To **what extent** do we want to **reduce** it ?
  - What is the **role** of numerical mixing in the **"good" performances** of numerical models ?
  - Is numerical mixing **making up** for a **lack** of physical mixing ?



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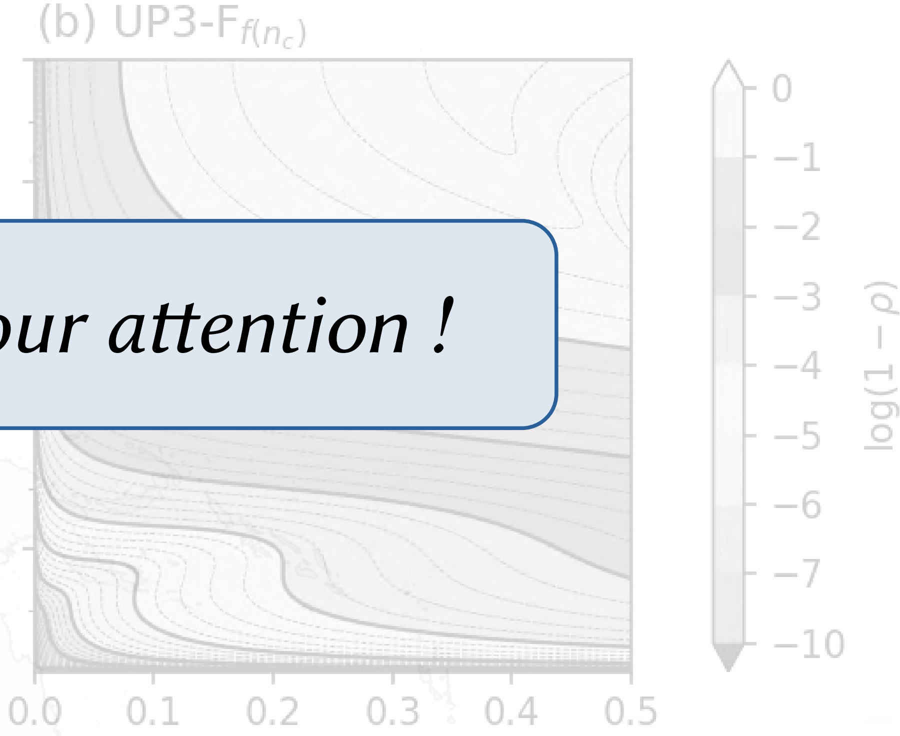

Thank you for your attention !