Stochastic parameterization of general non-hydrostatic processes with a focus on deep ocean convection

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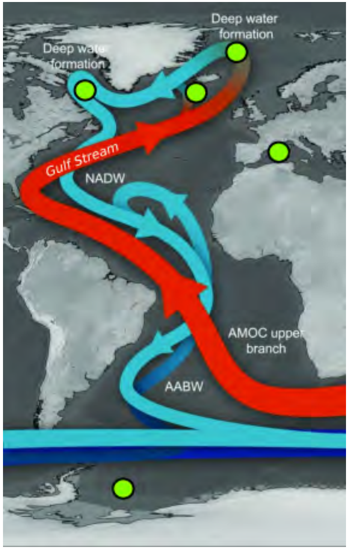
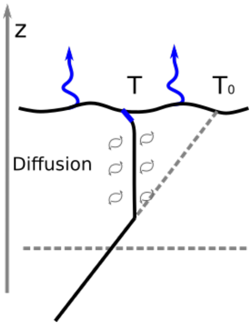


Figure 1: Schematic of AMOC (Crivalleri, 2018).

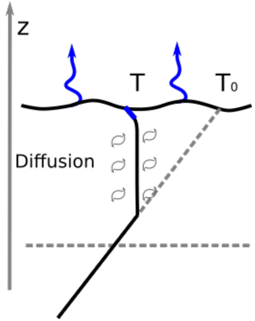
- Deep ocean convection sites are very localized in space:
  - Labrador Sea
  - Greenland Sea
  - Western Mediterranean
  - Weddell sea
- Control **deep water formation** rate
- Plays a key role in **large scale** ocean circulation, e.g. the AMOC



- 'Traditional' vertical mixing schemes are based on down-gradient formulation

$$\overline{w'\Psi'} = -\mathcal{K}_z \partial_z \overline{\Psi} \quad (1)$$

→ Local approach, limited capabilities for convective events
(Deardorff, 1966; Schmitt, 2007)



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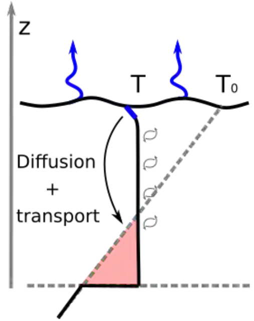


- Eddy-Diffusivity-Mass-Flux (EDMF) param. have shown better capabilities
(e.g. Hourdin et al., 2002; Soares et al., 2004; Suselj et al., 2019; Giordani et al., 2020):

$$\overline{w'\Psi'} = \underbrace{-\mathcal{K}_z \partial_z \overline{\Psi}}_{diff.} \underbrace{-a_p w_p (\Psi_p - \Psi)}_{mass\ flux} \qquad (2)$$

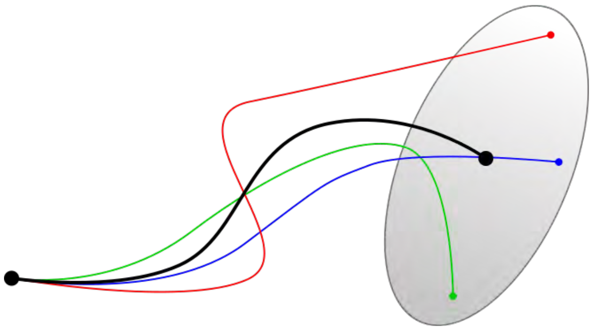
- Accounts for **non-local transport** associated with coherent structures

Outline

- Stochastic approach: The Location Uncertainty
- Free convection Large Eddy Simulation (LES)
- A 1D vertical application: The temperature equation
- A 3D application: The stochastic quasi-nonhydrostatic (SQ-NH) momentum equation
- Conclusions and discussion

The stochastic Lagrangian particle trajectory $\boldsymbol{X}_t$ is defined as:

$$\underbrace{\mathrm{d}\boldsymbol{X}_t}_{=\boldsymbol{X}_{t+\mathrm{d}t}-\boldsymbol{X}_t} = \underbrace{\boldsymbol{u}(\boldsymbol{X}_t, t)\mathrm{d}t}_{\text{resolved}} + \underbrace{\boldsymbol{\sigma}(\boldsymbol{X}_t, t)\mathrm{d}\boldsymbol{B}_t}_{\text{uncertainties}} \tag{3}$$



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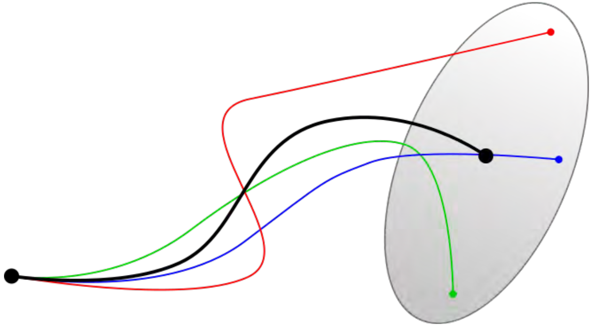
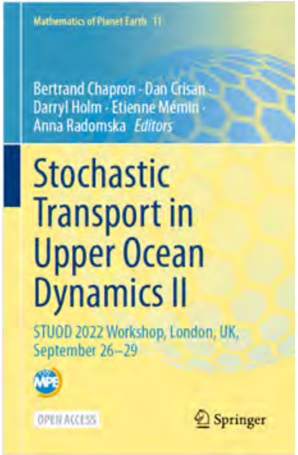


Through **stochastic calculus** (Itô-Wentzell),
the stochastic transport operator $\mathbb{D}_t$ for a stochastic (semi-martingal) process q is defined as:

$$\mathbb{D}_t q \triangleq \underbrace{\mathrm{d}_t q}_{\text{time increment}} + \underbrace{\left(\underbrace{\left(\boldsymbol{u} - \frac{1}{2}\nabla\cdot\boldsymbol{a} + \boldsymbol{\sigma}^T(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\right)}_{\boldsymbol{u}^\star}\cdot\boldsymbol{\nabla}\right)}_{\text{modified adv}} q\mathrm{d}t + \underbrace{\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t\cdot\boldsymbol{\nabla}q}_{\text{sto. adv}} - \underbrace{\frac{1}{2}\boldsymbol{\nabla}\cdot(\boldsymbol{a}\boldsymbol{\nabla}q)\,\mathrm{d}t}_{\text{sto. diff.}} \tag{4}$$

Recent generalization to compressible Navier–Stokes equations (Tissot et al., 2022)



By accounting for source terms in budget equation

$$\mathrm{d}\int_{\Omega(t)} q\mathrm{d}\boldsymbol{x} = \int_{\Omega(t)} \left(Q_t \mathrm{d}t + \boldsymbol{Q}_{\boldsymbol{\sigma}} \cdot \mathrm{d}\boldsymbol{B}_t\right) \mathrm{d}\boldsymbol{x}. \tag{5}$$

we arrived at the generalized Stochastic Reynolds Transport Theorem (SRTT):

$$\mathrm{d}_t q + \boldsymbol{\nabla} \cdot \left(\left((\boldsymbol{u} - \frac{1}{2}\nabla \cdot \boldsymbol{a})\mathrm{d}t + \boldsymbol{\sigma}_t \mathrm{d}\boldsymbol{B}_t\right) q\right) + \boldsymbol{\nabla} \cdot (\boldsymbol{\sigma}_t \boldsymbol{Q}_\sigma)\mathrm{d}t - \frac{1}{2}\boldsymbol{\nabla} \cdot (\boldsymbol{a}\boldsymbol{\nabla} q)\mathrm{d}t = Q_t \mathrm{d}t + \boldsymbol{Q}_\sigma \cdot \mathrm{d}\boldsymbol{B}_t \tag{6}$$

Transport equation for temperature T is obtained from conservation of total energy (i.e. SRTT(ρE))

$$\rho E = \rho \left(e + \frac{1}{2}||\boldsymbol{u}||^2 + gz \right), \quad (7)$$

with $e = \frac{T}{\gamma}$ the internal energy of the system, leading to:

$$\frac{\rho}{\gamma}\mathbb{D}_t T + A_T + A_u = P_t + P_\sigma + D_t + D_\sigma + V_t + V_\sigma \quad (8)$$

- A_T: Work of stochastic heat fluxes on stochastic flow
- A_u: Covariance of stochastic pressure
- P_t: 'Resolved' Compression/dilation effects
- P_σ: 'Stochastic' Compression/dilation effects
- D_t: 'Resolved' *drift work*
- D_σ: 'Stochastic' *drift work*
- V_t: Viscous contribution
- V_σ: Viscous contribution

Drift works D_t and D_σ can be related to *baropycnal work* present in compressible Large Eddy Simulation (Aluie, 2013).

→ Recover stochastic incompressible NS and HPE through low Mach and Boussinesq approximations, respectively, with possible **intermediate levels of approximation**.

Idealized Large Eddy Simulation (LES) of free convection with CROCO (Debreu et al., 2012)

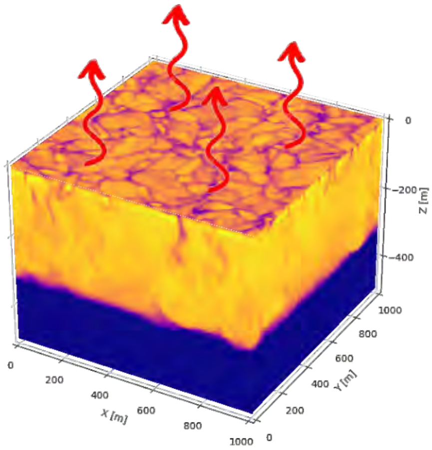


Figure 2: Snapshot after 72 hours

- **Physics**:
  - $N^2 = 2 \times 10^{-6}$ s^{-2}
  - $Q_{net} = 500$ W m^{-2}
  - 3 days long run
  - Linear EOS: $\rho = \rho_0(1 - \alpha_T T)$
  - No rotation: $f = 0$
- **Numerics**:
  - Domain size: 1 km x 1 km x 500m
  - Isotropic 10 m resolution
  - doubly periodic
  - Non-hydrostatic, non-Boussinesq (NBQ; Auclaire et al., 2018)
  - WENO5 advective scheme, but C2 for vert. adv. of temp.
  - 'Local' EVD for static instabilities

A 1D vertical application:
The temperature equation

Deterministic case

In our doubly periodic setting, horizontally averaged temperature equation reduces to

$$\partial_t \overline{T} = -\partial_z \left(\overline{w'T'} + \overline{wT_{sgs}} \right) \tag{9}$$

then

$$\overline{T}^{n+1} = \overline{T}^{n} - \underbrace{\Delta t}_{=10\text{ min}} \partial_z \left(\overline{w'T'} + \overline{wT_{sgs}} \right)^n \tag{10}$$

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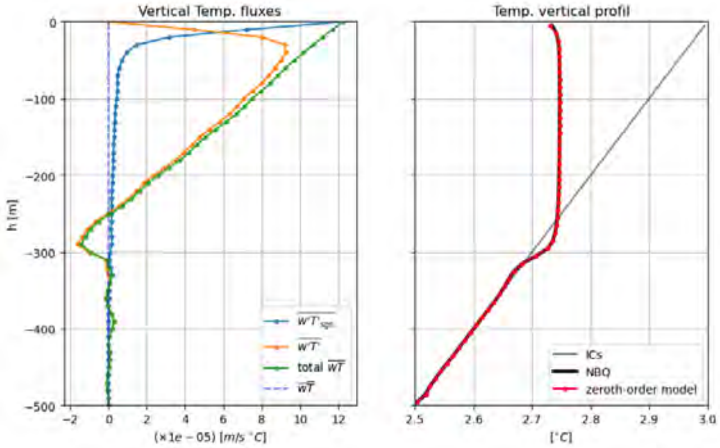

Stochastic case

In LU, transport equation for temperature reads:

$$\mathrm{d}_t\overline{T} = -(w^* - w)\partial_z\overline{T}\mathrm{d}t - \overline{\sigma \mathrm{d}B_t^{(z)}\partial_z T} + \frac{1}{2}\partial_z(a_{zz}\partial_z\overline{T})\mathrm{d}t + \text{others} \tag{11}$$

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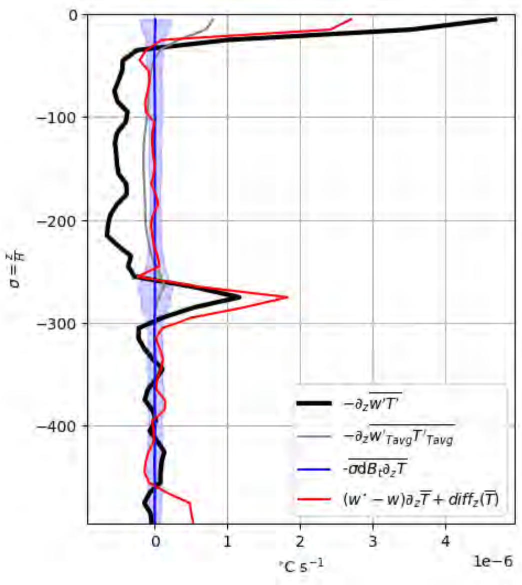


Stochastic transport of $\overline{T}$:

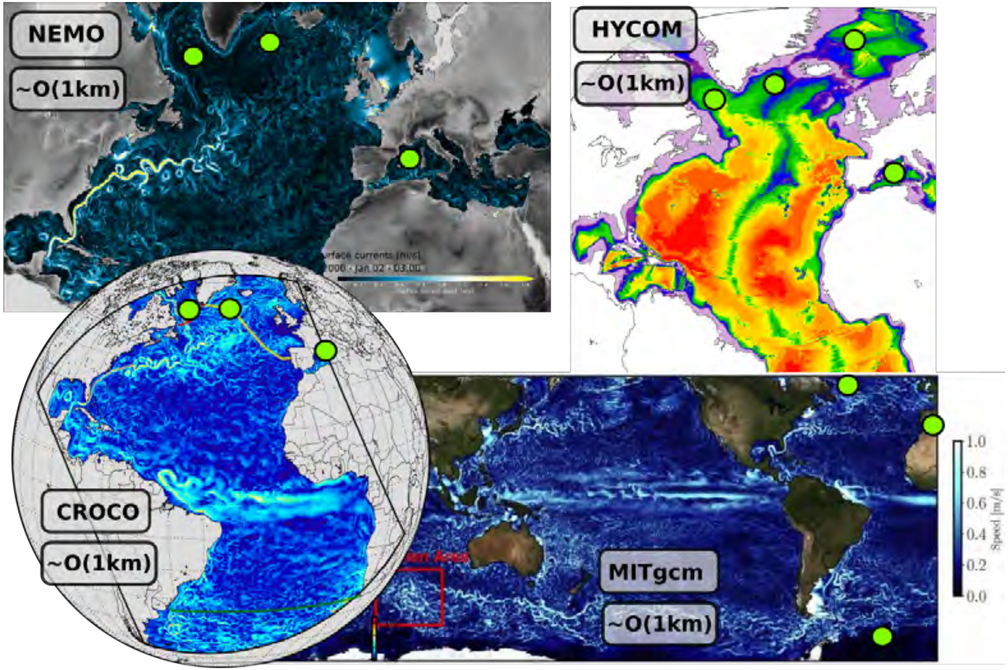
$$\overline{T}^{n+1} = \overline{T}^n - \Delta t \; RHS_{sto} \quad (12)$$

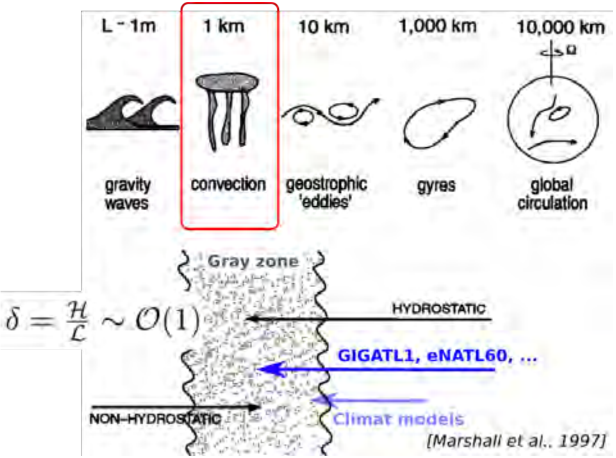
with:

- stochastic vert. adv. (blue)
- modified vert. adv + vert. diff (red)
  → **both contribute equally**
- $\Delta t = 8h$
- Deterministic estimates
  based on time averaged w ant T (gray)

A 3D application:
The stochastic quasi-nonhydrostatic (SQ-NH) momentum equation

Vertical mixing closure scheme are **1D vertical scheme**, i.e. no horizontal interactions between grid cells.
→ But kilometric-resolution simulations at basin/global scales are now emerging





Obj. 1: Re-introduce vertical dynamics without a full Non-Hydrostatic model
(e.g. Klingbeil & Burchard, 2013; Garreau, 2021)

Obj. 2: Use stochastic modelling

Stochastic quasi-nonhydrostatic (SQ-NH) model

Inspired by direct non-hydrostatic pressure correction method of Klingbeil & Burchard (2013) and Garreau (2021):

$$\partial_t u + \boldsymbol{\nabla}.\left(\boldsymbol{u} u\right) - f v = -\frac{1}{\rho_0}\partial_x\left(p + \qquad\right) + \mathcal{D}_u \tag{13a}$$

$$\partial_t v + \boldsymbol{\nabla}.\left(\boldsymbol{u} v\right) + f u = -\frac{1}{\rho_0}\partial_y\left(p + \qquad\right) + \mathcal{D}_v \tag{13b}$$

$$= -\frac{1}{\rho_0}\partial_z\left(p + \qquad\right)\mathrm{d}t - b\mathrm{d}t \tag{13c}$$

Vertical velocities are diagnostic through continuity

$$w(z) = -\int_{-H}^{\eta}(\partial_x u + \partial_y v)dz', \tag{14}$$

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$$\partial_t u + \boldsymbol{\nabla}.(\boldsymbol{u}u) - fv = -\frac{1}{\rho_0}\partial_x(p + p_{nh}) + \mathcal{D}_u \tag{13a}$$

$$\partial_t v + \boldsymbol{\nabla}.(\boldsymbol{u}v) + fu = -\frac{1}{\rho_0}\partial_y(p + p_{nh}) + \mathcal{D}_v \tag{13b}$$

$$(-\frac{1}{2}\boldsymbol{\nabla}\cdot\boldsymbol{a})\cdot\boldsymbol{\nabla}w\mathrm{d}t - \frac{1}{2}\nabla\cdot(\boldsymbol{a}\boldsymbol{\nabla}w)\mathrm{d}t = -\frac{1}{\rho_0}\partial_z(p + p_{nh})\,\mathrm{d}t - b\mathrm{d}t \tag{13c}$$

Vertical velocities are diagnostic through continuity

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Purely isotropic, homogeneous horizontal noise:

$$-(\frac{1}{2}\boldsymbol{\nabla}\cdot\boldsymbol{a})\cdot\boldsymbol{\nabla}w\mathrm{d}t - \frac{1}{2}\nabla\cdot(\boldsymbol{a}\boldsymbol{\nabla}w)\mathrm{d}t \rightarrow \frac{\nu_h}{2}\nabla_h^2 w \tag{15}$$

Stochastic quasi-nonhydrostatic (SQ-NH) model

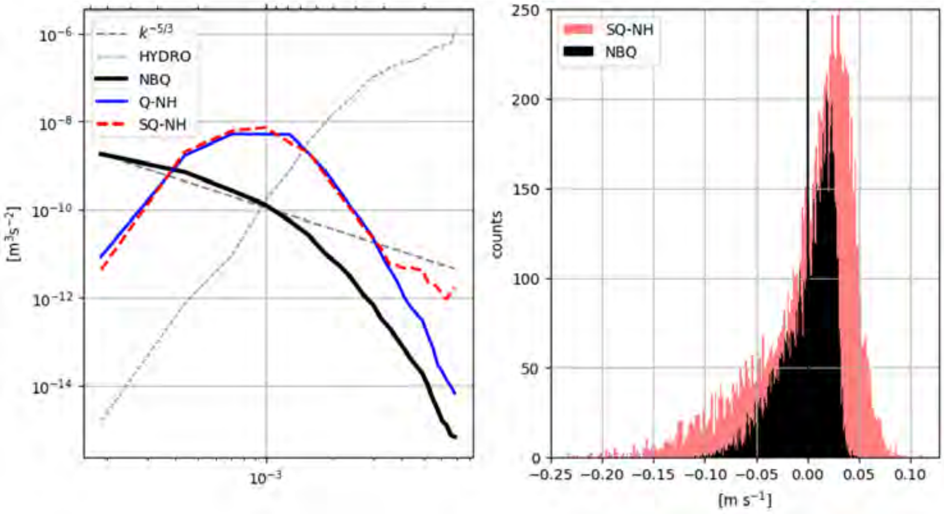

Figure 3: KE power spectra (left) and PDF of vertical velocities (right) for the reference **NBQ** and the stochastic quasi-nonhydrostatic **SQ-NH** simulations.

Conclusions

- Formalized stochastic compressible Navier–Stokes equations under Location Uncertainty (Tissot et al., 2022)
- Preliminary results in a 1D vertical application for the temperature equation and a in 3D application for momentum equation
- Implementation of a stochastic non-hydrostatic pressure correction in CROCO (Jamet et al., 2022)

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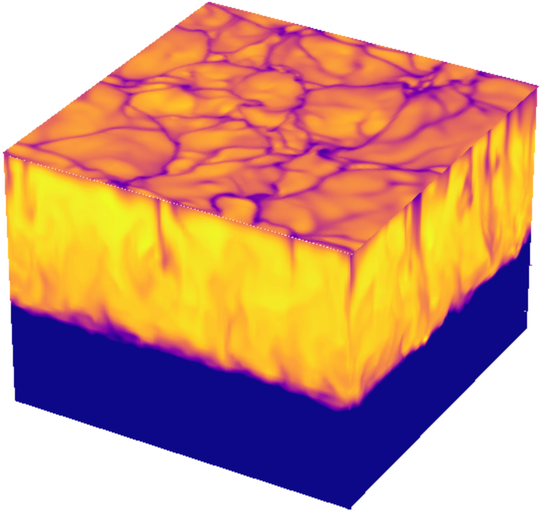
\+ Highlight the potential of stochastic modeling for penetrative convection in the 1D vertical application

\+ Partially recover spatial organization of convective plumes in the 3D application (but large scale energy is still missing ...)

– stochastic NH pressure correction inhibits penetrative convection :/ (similar in the deterministic case of Garreau (2021))



Thank you!

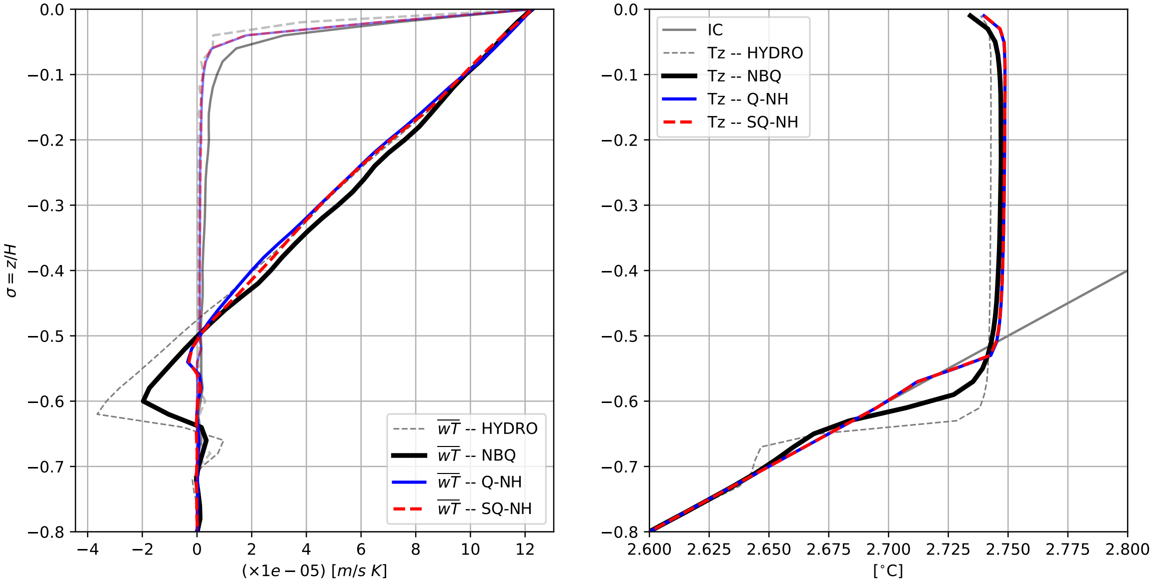
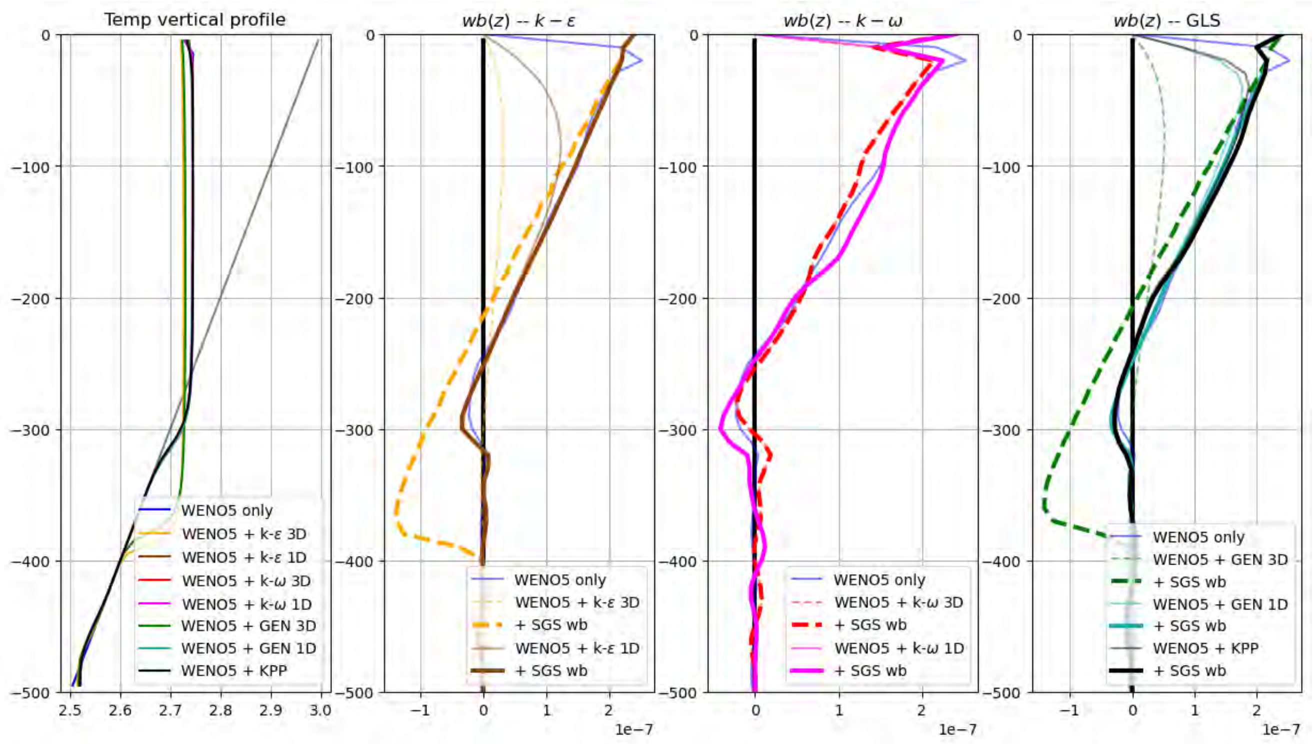


Figure 4: Vertical temperature fluxes (left), and vertical profile of temperature (right) after 3 days of simulation.

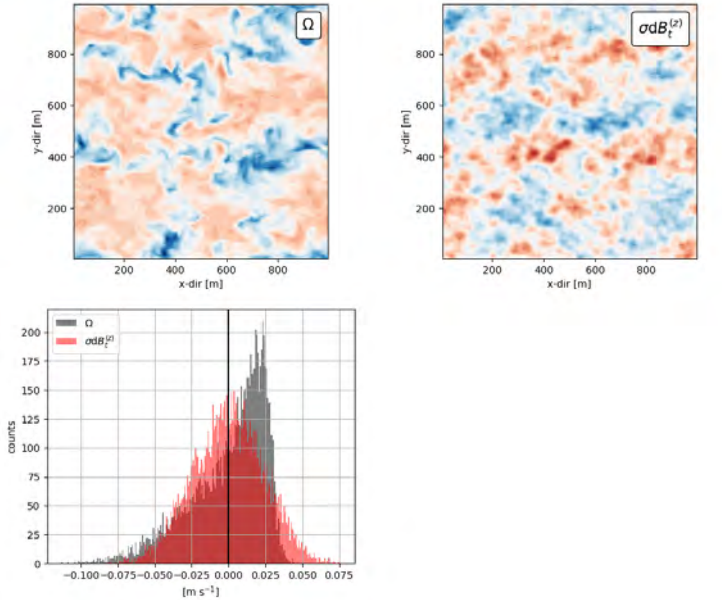

Temp vertical profile
$wb(z)$ -- $k-\varepsilon$
$wb(z)$ -- $k-\omega$
$wb(z)$ -- GLS
0
−100
−200
−300
−400
−500
2.5 2.6 2.7 2.8 2.9 3.0
−1 0 1 2
1e−7
0 1 2
WENO5 only
WENO5 + k-ε 3D
WENO5 + k-ε 1D
WENO5 + k-ω 3D
WENO5 + k-ω 1D
WENO5 + GEN 3D
WENO5 + GEN 1D
WENO5 + KPP
+ SGS wb

$$
\begin{aligned}
&\frac{\rho}{\gamma}\mathbb{D}_t T + \underbrace{\frac{\rho}{\gamma}\sum_k \mathrm{d}_t \left\langle \int_0^{\cdot} \boldsymbol{\sigma}_s \mathrm{d}\boldsymbol{B}_s{}^k, \int_0^{\cdot} \frac{\partial}{\partial x_k}\left(\boldsymbol{F}_\sigma^T \cdot \mathrm{d}\boldsymbol{B}_s\right) \right\rangle_t}_{A_T} \\
&+ \underbrace{\frac{\rho}{2}\sum_i \mathrm{d}_t \left\langle \int_0^{\cdot} \boldsymbol{F}_\sigma^{u_i} \cdot \mathrm{d}\boldsymbol{B}_s, \int_0^{\cdot} \boldsymbol{F}_\sigma^{u_i} \cdot \mathrm{d}\boldsymbol{B}_s \right\rangle_t}_{A_u} \\
&= \underbrace{-p\nabla \cdot (\boldsymbol{u}^\star \mathrm{d}t + \boldsymbol{\sigma}_t \mathrm{d}\boldsymbol{B}_t)}_{P_t} \underbrace{-\mathrm{d}p_t^\sigma \nabla \cdot \boldsymbol{u}^\star}_{P_\sigma} \\
&+ \underbrace{\frac{1}{Re}\boldsymbol{\tau}(\boldsymbol{u}) : \nabla\left(\boldsymbol{u}^\star \mathrm{d}t + \boldsymbol{\sigma}_t \mathrm{d}\boldsymbol{B}_t\right)}_{V_t} + \underbrace{\frac{1}{Re}\boldsymbol{\tau}(\boldsymbol{\sigma}_t \mathrm{d}\boldsymbol{B}_t) : \nabla \boldsymbol{u}^\star}_{V_\sigma} \\
&+ \underbrace{\left((\boldsymbol{u}^\star - \boldsymbol{u})\,\mathrm{d}t + \boldsymbol{\sigma}_t \mathrm{d}\boldsymbol{B}_t\right) \cdot \left(-\nabla p + \rho \boldsymbol{g} + \frac{1}{Re}\nabla \cdot \boldsymbol{\tau}(\boldsymbol{u})\right)}_{D_t} \\
&+ \underbrace{(\boldsymbol{u}^\star - \boldsymbol{u}) \cdot \left(-\nabla \mathrm{d}p_t^\sigma + \frac{1}{Re}\nabla \cdot \boldsymbol{\tau}(\boldsymbol{\sigma}_t \mathrm{d}\boldsymbol{B}_t)\right)}_{D_\sigma} \\
&+ \sum_i \frac{1}{ReSc_i}\nabla \cdot \left((\rho Y_i c_{v,i} T)\nabla Y_i\right)\mathrm{d}t + \frac{\rho}{RePr}\nabla \cdot (\nabla T)\,\mathrm{d}t.
\end{aligned}
$$

Stochastic case

Noise construction from LES numerical simulations:

$$\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t(x,y,,z,t) = \sum_n \hat{\boldsymbol{b}}_n(z,t)\boldsymbol{\phi}_n(x,y)\mathrm{d}B_t^{(n)} \tag{16}$$

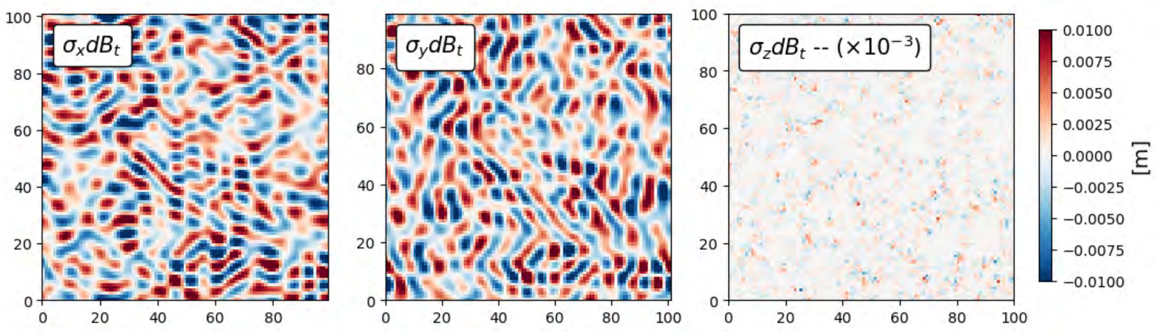


Noise structure:
Horizontally homogeneous, isotropic and white noise → random plane waves

$$\boldsymbol{\sigma} \mathrm{d}\boldsymbol{B}_t^H = \boldsymbol{\nabla}_H^{\perp} \theta \tag{17}$$

with

$$\theta = \sum_n \left(e^{2\pi i \mathbf{k}^{(n)} \cdot \mathbf{x}} \, \mathrm{d}\boldsymbol{B}_t^{(n)} \right) * F(z, t). \tag{18}$$



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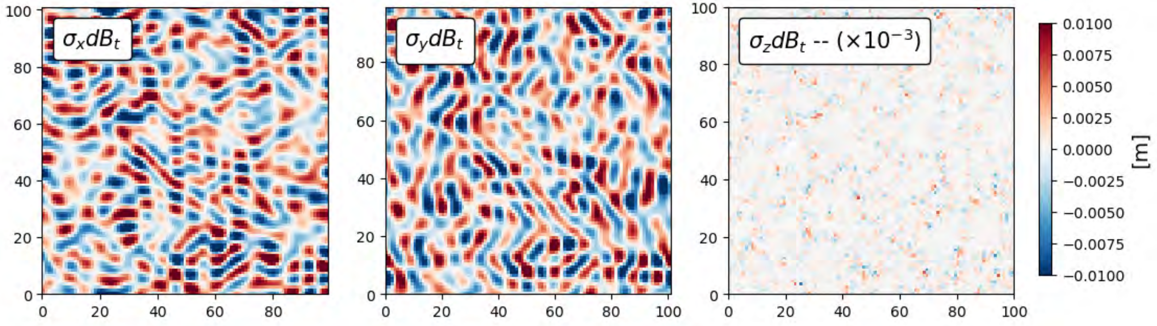

$$\boldsymbol{a}(\boldsymbol{x}, t) = \boldsymbol{\sigma}\boldsymbol{\sigma}^T \quad ; \quad -(\frac{1}{2}\boldsymbol{\nabla} \cdot \boldsymbol{a}) \cdot \boldsymbol{\nabla} w \mathrm{d}t - \frac{1}{2} \nabla \cdot (\boldsymbol{a} \boldsymbol{\nabla} w) \mathrm{d}t \rightarrow \frac{\nu_h}{2} \nabla_h^2 w \tag{19}$$