From Virtual Navier-Stokes Flows to Numerical Atmosphere & Ocean Models or *A Discrete Model Hierarchy*

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CLIVAR & COMMODORE workshop, NCAR, September 9

Overview

I will discuss the following topics

1. Incompressible Dynamics *($\sim$ ocean)*
2. Compressible Dynamics *($\sim$ atmosphere)*
3. Singular Limits *(relation between different equations)*
4. Lessons learned

Focus on **finite-dimensional** setup and **nonhydrostatic dynamics**

Starting Point: Primitive Equations - Hydrostatic and Boussinesq

Velocity field: $\mathbf{v} = (v_h, w)$, horizontal velocity v_h, vertical velocity w

$$\partial_t v_h + \omega_z \vec{e}_z \times v_h + \frac{\nabla_h |v_h|^2}{2} + w \partial_z v_h + \frac{1}{\rho_0} \nabla_h p - \mathcal{D} v_h = 0$$

$$\partial_z p = -\rho g$$

$$\partial_t \eta + div_h \int_{-B}^{\eta} v \, dz = 0$$

$$div_h \, v_h + \partial_z w = 0$$

$$\partial_t C + div(C\mathbf{v}) - div(\mathbb{K}^C \nabla C) = 0$$

$$\rho = F_{eos}(p, T, S),$$

Incompressible Dynamics: From Hydrostatic to Nonhydrostatic

Starting Point: Primitive Equations - Hydrostatic and Boussinesq

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$$\rho = F_{eos}(p, T, S),$$

This is the Hydrostatic Euler Equation.
How can we make it NonHydrostatic ?

Route A to Nonhydrostatic Euler: add w-eq to hydrostatic eqs.

$$\partial_t v_h + \omega_z \vec{e}_z \times v_h + w\partial_z v_h + \nabla_h \left(p + \frac{|v_h|^2}{2}\right) = 0,$$

$$\partial_t w + (v, w) \cdot \nabla w + \partial_z p = 0,$$

Route B to Nonhydrostatic Euler: 3D vector-invariant

$$\partial_t \mathbf{v} + \omega \times \mathbf{v} + \nabla \left(p + \frac{|\mathbf{v}|^2}{2}\right) = 0 \qquad (\mathbf{v}, \omega \textit{ are 3D vector fields})$$

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- **Route A:** easy to implement, breaks beauty of Euler equation. *(no consistent vorticity eq., energetics presumably impossible...)*
- **Route B** challenge is discrete exterior product $\omega \times \mathbf{v}$

Strategy: we go for Route B

1. Focus on inviscid case and get conservation properties
2. Incorporate dissipation via *explicit dissipation, upwind-biased . . .*

Incompressible Euler Equations

$$\partial_t \mathbf{v} + \omega \times \mathbf{v} + \nabla\left(p + \frac{|\mathbf{v}|^2}{2}\right) = 0, \quad div\mathbf{v} = 0$$

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Advection - Continuous Cross Product

$$\text{horiz. v-equation: } (\omega \times \mathbf{v})|_h = \begin{pmatrix} \omega_y v_z - \omega_z v_y \\ \omega_z v_x - \omega_x v_z \end{pmatrix},$$

$$\text{vert. v-equation: } (\omega \times \mathbf{v})|_v = \omega_h \cdot \mathbf{v}_h^\perp = \left(\omega_x v_y - \omega_y v_x \right).$$

- blue/Hydrostatic: Cross-product terms with vertical vorticity ω_z
- red/Nonhydrostatic:Cross-product terms with horiz. vorticity ω_h
- $\rightarrow$ we have blue we need red

We need 3D vorticity vector (ω_h, ω_z) *and construct missing* ω_h *via Stokes Theorem*

Horizontal component of vorticity vector - continuous

$$\omega_h := curl_h \mathbf{v} = \begin{pmatrix} \omega_x \\ \omega_y \end{pmatrix} = \begin{pmatrix} \partial_y v_z - \partial_z v_y \\ \partial_z v_x - \partial_x v_z \end{pmatrix}$$

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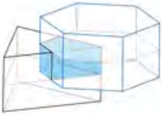
Three Observations

1. Prismatic grid: 2D horizontal $\times$ 1D vertical
2. Dual prism is shifted horizontally and vertically
3. Vertical faces are rectangles !

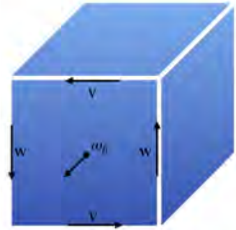


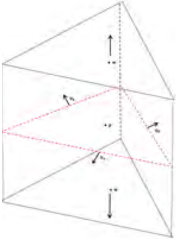
Horizontal component of vorticity vector $\omega_{\partial\hat{P}}$ via Stokes

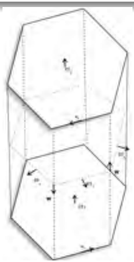
$$\omega_h \underbrace{\rightsquigarrow}_{\text{by Stokes}} \mathbf{curl}_h u_{\partial P} := w_{K,k+1/2} + v_{e,k} - w_{L,k+1/2} - v_{e,k+1}$$

This defines

- discrete 3D curl-operator: $\mathbf{curl} u_{\partial P} = (\mathbf{curl}_h u_{\partial P}, \mathbf{curl}_v u_{\partial P})$
- vorticity vector at faces of dual prism $\omega_{\partial\hat{P}} := (\omega_h, \omega_v)$



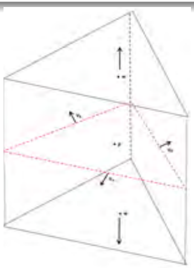


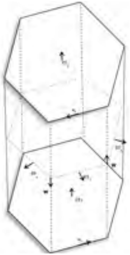


$$\begin{pmatrix} \omega_y v_z - \omega_z v_y \\ \omega_z v_x - \omega_x v_z \\ \omega_y v_y - \omega_y v_x \end{pmatrix} \rightsquigarrow \omega_{\partial \hat{P}} \star u_{\partial P} := \begin{pmatrix} \hat{\mathcal{P}}_h^\dagger(\omega_z \hat{\mathcal{P}}_h v) - \mathcal{P}_z \mathcal{P}^T(w \tilde{\mathcal{P}}_h \omega_h) \\ \tilde{\mathcal{P}}_h \omega_h \cdot \mathcal{P}\mathcal{P}_z^T v \end{pmatrix}$$

the secret sauce

- $\mathcal{P}, \hat{\mathcal{P}}, \tilde{\mathcal{P}}$ are *Hilbert space compatible reconstructions*





Kernel of Differential Operators

1. $\mathbf{grad} p = 0$ if and only if p is constant
2. $\mathbf{curl}\, v = 0$ if and only if $v = \mathbf{grad}\, p$
3. $\mathbf{div}\, v = 0$ if and only if $v = \mathbf{curl}^T u$.

Discrete Biot-Savart:

From given $\omega \in \mathcal{H}_{\hat{V}}$ the velocity $u_{\partial P} \in \mathcal{H}_{\partial P}$ with $\mathbf{div} \mathcal{M}\, u_{\partial P} = 0$, is recovered by solving Laplace equation

$$\mathbf{div} \mathcal{M} \mathbf{grad}\, u_{\partial P} = \mathbf{curl}^T \omega.$$

Cont. Velocity Space & Discrete Degrees of Freedom on Prism Q

- $\mathbb{F}(Q) := \{f \in H_{div}(Q) \cap H_{rot}(Q) : \; div\, f \in \mathbb{P}_0(Q), curl\, f = 0, \\ f|_e \cdot \mathbf{n}_e \in \mathbb{P}_0(Q) \, \forall e \in \partial Q\},$
- $dof_{\mathbb{F}(Q)}(f) := \Pi f := \frac{1}{|e|} \int_e f \cdot \mathbf{n}_e \, ds, \quad \forall e \in \partial Q.$

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Theorem

Discrete DoF's above are unisolvent, i.e. they characterize uniquely the respective continuous virtual element space.

Reconstructions of disc. DoF via local PDEs

Given discrete velocity dof's $v_e \in \partial Q$. Define continuous function $\tilde{v} := \mathcal{P}v$ on Q as solution of local div-curl problem

$$\begin{aligned} div\,\tilde{v} &= \mathbf{div}\,v, && \text{on } Q, \\ curl\,\tilde{v} &= 0, && \text{on } Q, \\ \tilde{v} \cdot \mathbf{n}_e &= v_e && \text{on } \partial Q. \end{aligned}$$

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Scalar Product on Discrete Velocity Space in Terms of Reconstructions

$$\langle u, v\rangle_{\mathbb{F}(Q)} := \int_Q \mathcal{P}u\cdot\mathcal{P}v\,dx$$

$$\int_\Omega \mathcal{P}u\cdot\mathcal{P}v\,dx = \sum_{Q\in\mathcal{C}} |Q|\mathcal{P}u_Q\cdot\mathcal{P}v_Q,$$

Pressure & Vorticity

For pressure and vorticity spaces similar scalar products via reconstructions via local div-curl PDEs.

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Fundamental Lemma on Reconstructions

Let $\mathcal{P} : v_e \to \mathcal{P}v \in \mathbb{F}(Q)$ be a reconstruction such that

- $\mathcal{P}$ is the right-inverse of projection $\Pi f := \frac{1}{|e|} \int_e f \cdot \mathbf{n}_e \, d$
- $\mathcal{P}$ is first-order accurate
- $\mathcal{P}$ commutes with continuous differential operators *grad*, *div*, *curl*
- Reconstructed functions are orthogonal to linear polynomials on Q with zero mean
- $\mathcal{P}$ has a local stencil

Then it holds $\int_Q \mathcal{P}v \cdot \mathbf{e}_i dx = \sum_{e \in \partial Q} v_e |e| (\mathbf{x}_e - \mathbf{x}_Q) \cdot \mathbf{e}_i$.

(Analogous results for $\hat{\mathcal{P}}, \tilde{\mathcal{P}}$)

Reconstructions

- Div-Curl- PDE are actually never solved.
- Reconstructions have an explicit & computable form.
- We need three Reconstructions
  - $\mathcal{P}$: face dof $\rightarrow$ inside primal 3D prism
  - $\hat{\mathcal{P}}$: face dof $\rightarrow$ inside 3D dual prism
  - $\tilde{\mathcal{P}}_h$: edge dof $\rightarrow$ inside 2D primal cell
  - $\mathcal{M} := \mathcal{P}^T \mathcal{P}$

End of Numerical Disgression - Back to Euler Equations

Incompressible Euler

- $\langle \frac{d}{dt}\mathcal{M}u_{\partial P}, \phi \rangle_{\mathcal{H}_{\partial P}} + \langle \omega_{\partial \hat{P}} \star u_{\partial P}, \phi \rangle_{\mathcal{H}_{\partial P}}$ $+ \langle \mathcal{M}\mathbf{grad}(p + \frac{|\mathcal{P}u_{\partial P}|^2_{\mathbb{R}^3}}{2}), \phi \rangle_{\mathcal{H}_{\partial P}} = 0, \ \forall \phi \in \mathcal{H}_{\partial P},$
- $\mathbf{div}\mathcal{M}u_{\partial P} = 0.$

Incompressible Euler

- $$\left\langle \frac{d}{dt}\mathcal{M}u_{\partial P}, \phi \right\rangle_{\mathcal{H}_{\partial P}} + \left\langle \omega_{\partial \hat{P}} \star u_{\partial P}, \phi \right\rangle_{\mathcal{H}_{\partial P}} + \left\langle \mathcal{M}\mathbf{grad}\left(p + \frac{|\mathcal{P}u_{\partial P}|^2_{\mathbb{R}^3}}{2}\right), \phi \right\rangle_{\mathcal{H}_{\partial P}} = 0, \ \forall \phi \in \mathcal{H}_{\partial P},$$
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Pressure Recovery

Pressure is recovered from $u_{\partial P} \in \mathcal{H}_{\partial P}$ by solving Laplace equation

$$\mathbf{div}\mathcal{M}\mathbf{grad}p = -\mathbf{div}\mathcal{M}\mathbf{grad}\left(\frac{|\mathcal{P}_3 u_{\partial P}|^2_{\mathbb{R}^3}}{2}\right) - \mathbf{div}\mathcal{L}(\omega_{\partial \hat{P}}, u_{\partial P})$$

Theorem (Well-Posedness of Semi-Discrete Euler Equations)

- *Let a time interval* $[0, T]$ *and initial conditions* $u_0 \in \mathcal{H}_{\partial P}^{div}$ *be given.*

Then there exist for $t \in [0, T]$ *a unique solution* $u_{\partial P}(t) \in \mathcal{H}_{\partial P}$ *of the discrete Euler equations.*

(Proof by Picard's theorem for ODE for short time and extension to long time via energy conservation.)

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Well-Posedness of Semi-Discrete Navier-Stokes

Proof translates to Navier-Stokes equations,
with dissipation given by $\mathcal{D}(v) := \mathbf{curl}^T(\nu \mathbf{curl} u_{\partial P})$.

Linear Momentum

Let $u_{\partial P} \in \mathcal{H}_{\partial P}^{divM}$ be a solution of the Euler equation. Then the linear momentum $\mathcal{I} := \left\langle \mathcal{P}^T \mathcal{P} u_{\partial P}, 1 \right\rangle_{\mathbf{H}_{\mathcal{P}}}$ satisfies

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Angular Momentum $L := \vec{x} \times \vec{u}$

- Question: How to define $L := \vec{x} \times \vec{u}$ for staggered $\vec{u} = (v_h, w)$?
- Observation: Vector product $\vec{\omega} \times \vec{u} \sim \omega_{\partial \hat{P}} \star u_{\partial P}$

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Angular Momentum - Definition and Conservation

Define discrete angular momentum:

$$\ell(u_{\partial P}) := \vec{x} \star u_{\partial P},$$

where $\vec{x}$ is coordinate vector of the position of vorticity at dual prism faces. Then

$$\frac{d}{dt}\left\langle \ell(u_{\partial P}), 1 \right\rangle = 0$$

Theorem (Energy Conservation)

The solution $u_{\partial P}(t) \in \mathcal{H}^{div}_{\partial P}$ of the discrete incompressible Euler equations conserves kinetic energy:

$$\frac{d}{dt} E^{kin}(t) = 0, \quad E^{kin}(t) := ||\mathcal{P}_3 u_{\partial P}(t)^2||_{\mathbf{H}_{\mathcal{P}}}$$

2D: *Energy and enstrophy conservation.*

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Helicity: Inner Product of Velocity & Vorticity

$\mathcal{H} := \int_{\Omega} \mathbf{v} \cdot \omega \, dx$ | $\mathbf{H} := \left\langle \hat{\mathcal{P}} v \, \omega_z, 1 \right\rangle_{\mathcal{H}_{\hat{V}}} + \left\langle w \tilde{\mathcal{P}}_h \omega_h, 1 \right\rangle_{\mathcal{H}_P}$

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Theorem (Helicity Conservation)

The solution $u_{\partial P_k}(t) \in \mathcal{H}^{div}_{\partial P}$ *of the discrete* **3D** *incompressible Euler equations conserves helicity:*

$$d_t \mathbf{H} = 0.$$

(Proof by combining equations for vorticity and velocity and suitable test functions in discrete weak form.)

Time stepping - Implicit

$$\left\langle \frac{\mathcal{M}(u_{\partial P}^{n+1} - u_{\partial P}^{n})}{\Delta t}, \phi \right\rangle_{\mathcal{H}_{\partial P}} + \left\langle \omega_{\partial \hat{P}}^{n+1/2} \star u_{\partial P}^{n+1/2}, \phi \right\rangle_{\mathcal{H}_{\partial P}}$$

$$+ \left\langle \mathcal{M}\mathbf{grad}\left(p^{n+1/2} + \frac{|\mathcal{P} u_{\partial P}^{n+1/2}|_{\mathbb{R}^3}^2}{2}\right), \phi \right\rangle_{\mathcal{H}_{\partial P}} = \left\langle f^{n+1/2}, \phi \right\rangle_{\mathcal{H}_{\partial P}},$$

$$\mathbf{div}\mathcal{M} u_{\partial P}^{n+1} = 0,$$

where $u_{\partial P}^{n+1/2} := \frac{1}{2}(u_{\partial P}^{n+1} + u_{\partial P}^{n})$

Theorem

• *Let $\Delta t > 0$ be the time step size and $u^0_{\partial P} in \mathcal{H}^{divM}_{\partial P}$ initial conditions. Then a unique solution exists $(u^n_{\partial P}, p^n)$ of incompressible Euler with the following properties:*

1. $(u^n_{\partial P}, p^n)$ *conserves global kinetic energy,* $E^{kin}(u^n_{\partial P}) = E^{kin}(u^0_{\partial P})$
2. $(u^n_{\partial P}, p^n)$ *conserves linear momentum* $\mathcal{I}(u^n_{\partial P}) = \mathcal{I}(u^0_{\partial P})$ *and angular momentum* $\ell(u^n_{\partial P}) = \ell(u^0_{\partial P})$
3. $(u^n_{\partial P}, p^n)$ *conserves vorticity* $\left\langle {\omega_{\partial \hat{P}}}^n, 1 \right\rangle_{\mathcal{H}_{\hat{V}}} = \left\langle {\omega_{\partial \hat{P}}}^0, 1 \right\rangle_{\mathcal{H}_{\hat{V}}}$.
4. $(u^n_{\partial P}, p^n)$ *conserves helicity* $\mathbf{H}(u^n_{\partial P}) = \mathbf{H}(u^0_{\partial P})$
5. $(u^n_{\partial P}, p^n)$ *is reversible in time*

(Proof: Schauder fix point theorem, differentiability of mapping for uniqueness. Conservation properties rely on implicit time stepping.)

Navier-Stokes Equations

Proof applies to Navier-Stokes, without conservation props.

Now allow the density to vary - but not to compress

Incompressible Euler-Boussinesq Equations

$$\partial_t \mathbf{v} + \omega \times \mathbf{v} + \nabla\big(p + \frac{|\mathbf{v}|^2}{2}\big) = g\rho\vec{e}_z, \quad div\mathbf{v} = 0$$

$$\partial_t \rho + div(\rho\mathbf{v}) = 0.$$

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Euler-Boussinesq

- $\big\langle \frac{d}{dt}\mathcal{M}u_{\partial P}, \phi\big\rangle_{\mathcal{H}_{\partial P}} + \big\langle \omega_{\partial\hat{P}} \star u_{\partial P}, \phi\big\rangle_{\mathcal{H}_{\partial P}}$
  $+ \big\langle \mathcal{M}\mathbf{grad}\big(p + \frac{|\mathcal{P}u_{\partial P}|^2_{\mathbb{R}^3}}{2}\big), \phi\big\rangle_{\mathcal{H}_{\partial P}} = \big\langle g\rho\vec{e}_z, \phi\big\rangle_{\mathcal{H}_P}$
- $\mathbf{div}\mathcal{M}u_{\partial P} = 0,$
- $\big\langle \partial_t \rho, \psi\big\rangle_{\mathcal{H}_P} + \big\langle \mathbf{div}(\mathcal{P}^T(\rho\mathcal{P}v)), \psi\big\rangle_{\mathcal{H}_P} = 0.$

Theorem (Well-Posedness of Semi-Discrete Euler-Boussinesq)

i) A unique solution $u_{\partial P}(t) \in \mathcal{H}^{div}_{\partial P}, \rho \in \mathcal{H}_P$ *to discrete Euler-Boussinesq equations exists*
ii) and it has the following properties

- **Energy Conservation:** *The sum of kinetic and potential energy is conserved*

$$\frac{d}{dt}(E^{kin} + E^{pot})(t) = 0, \quad E^{pot} := g\rho Q w$$

- **Helicity Conservation** *The solution* $u_{\partial P_k}(t) \in \mathcal{H}^{div}_{\partial P}$ *of the discrete* **3D** *incompressible Euler equations satisfies*

$$d_t\mathbf{H} = F(\Phi). \qquad (\Phi \text{ geopotential})$$

New Invariant: Potential Vorticity (Mass Weighted)

PV - Continuous:

$$\mathcal{PV} := \omega \cdot \nabla \rho$$

PV - Discrete: Inner Product of ω and $\mathbf{grad}\rho$

$$\mathbf{PV}(u_{\partial P})|_{\hat{P}} := \left\langle \omega_z \, \hat{\mathcal{P}}_h \mathbf{grad}\rho, 1 \right\rangle_{\mathbf{H}_{\partial \hat{P}=}} + \left\langle \omega_h \, \mathbf{D_z}\rho, 1 \right\rangle_{\mathcal{H}_{\partial \hat{P}||}}$$

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Potential Vorticity Conservation

Let $u_{\partial P} \in \mathcal{H}_{\partial P}^{divM}$ be a solution of the Euler-Boussinesq equations. Then

$$\frac{d}{dt} \langle \mathbf{PV}, 1 \rangle_{\mathcal{H}_{\hat{P}}} = 0.$$

Proof by combining equations for vorticity and for density gradient.

Theorem - Implicit Time stepping

Let $\Delta t > 0$ be the time step size. Let initial conditions ${u_{\partial P}}^0 \in \mathcal{H}_{\partial P}^{divM}$ be given. Then there exists a unique solution $(u_{\partial P}^n, p^n)$ of Euler-Bousinesq with the following properties:

1. $(u_{\partial P}^n, p^n)$ conserves global kinetic energy, $E^{kin}(u_{\partial P}^n) = E^{kin}(u_{\partial P}^0)$
2. $(u_{\partial P}^n, p^n)$ conserves linear momentum $\mathcal{I}(u_{\partial P}^n) = \mathcal{I}(u_{\partial P}^0)$
3. $(u_{\partial P}^n, p^n)$ conserves helicity $\mathbf{H}(u_{\partial P}^n) = \mathbf{H}(u_{\partial P}^0)$
4. $(u_{\partial P}^n, p^n)$ conserves the potential vorticity $\mathbf{PV}(u_{\partial P}^n) = \mathbf{PV}(u_{\partial P}^0)$

Small Aspect Ratio ϵ

- Thin domain: $\Omega_\epsilon = [-1,1]^2 \times [-\epsilon,\epsilon]$ transform into $\Omega := [-1,1]^3$
- Transformation:

$$\mathbf{v}_\epsilon(x,y,z,t) := \mathbf{v}(x,y,\epsilon z,t),, \quad p_\epsilon(x,y,z,t) := p(x,y,\epsilon z,t)$$
$$w_\epsilon(x,y,z,t) := \frac{1}{\epsilon} w(x,y,\epsilon z,t).$$

Scaled Euler Equations

$$\partial_t \mathbf{v}_\epsilon + (curl\mathbf{v}\epsilon \times \mathbf{v}_\epsilon)|_h + \nabla_h \frac{|\mathbf{v}_\epsilon|^2}{2} + \nabla_h p_\epsilon = 0,$$
$$\epsilon^2 \left\{ \partial_t w_\epsilon + (curl\mathbf{v}_\epsilon \times \mathbf{v}_\epsilon)|_v + \partial_z \frac{|\mathbf{v}_\epsilon|^2}{2} \right\} + \partial_z p_\epsilon = 0,$$
$$div\, \mathbf{v}_\epsilon + \partial_z w_\epsilon = 0.$$

Scaled Euler Equations $\mathbf{v}_\epsilon = (v_1, v_2, v_3), v_i = v_i(x, y, z, t)$

$$\partial_t \mathbf{v}_\epsilon + (curl \mathbf{v}_\epsilon \times \mathbf{v}_\epsilon)|_h + \nabla_h \frac{|\mathbf{v}_\epsilon|^2}{2} + \nabla_h p_\epsilon = 0,$$

$$\epsilon^2 \left\{ \partial_t w_\epsilon + (curl \mathbf{v}_\epsilon \times \mathbf{v}_\epsilon)|_v + \partial_z \frac{|\mathbf{v}_\epsilon|^2}{2} \right\} + \partial_z p_\epsilon = 0,$$

$$div\, \mathbf{v}_\epsilon + \partial_z w_\epsilon = 0.$$

Hydrostatic Euler Equations $\mathbf{v} = (v_1, v_2), v_i = v_i(x, y, z, t)$

$$\partial_t \mathbf{v} + (curl \mathbf{v} \times \mathbf{v})|_h + \nabla_h \frac{|\mathbf{v}|^2}{2} + \nabla_h p = 0,$$

$$\partial_z p = 0,$$

$$div\, \mathbf{v}_h + \partial_z w = 0.$$

What happens for $\epsilon \to 0$?

Discrete Scaled Euler Equations

- $\langle \frac{d}{dt}\mathcal{M}_h v^\epsilon + \omega_{\partial\hat{P}} \star u_{\partial P}|_h^{nh} + \mathcal{M}_h \mathrm{grad}_n(\frac{E_{kin}^{nh}}{2} + p^\epsilon), \phi_h\rangle_{\mathcal{H}_\mathcal{F}} = 0,$
- $\langle \epsilon^2 \left\{ \frac{d}{dt} w^\epsilon + \omega_{\partial\hat{P}} \star u_{\partial P}|_z^{nh} + \mathbf{D_z}(\frac{|\mathcal{P}u_{\partial P}{}^\epsilon|^2}{2}) \right\} + \mathbf{D_z} p^\epsilon, \phi_v\rangle_{\mathcal{H}_P} = 0,$
- $\mathbf{div}_h \mathcal{M}_h v^\epsilon + \mathbf{div}_v w^\epsilon = 0,$

Discrete Hydrostatic Euler Equations

- $\langle \frac{d}{dt}\mathcal{M}_h v + \omega_{\partial\hat{P}} \star u_{\partial P}|_h^{hyd} + \mathcal{M}_h \mathrm{grad}_n(p + \frac{E_{kin}^{hyd}}{2}), \phi_h\rangle_{\mathcal{H}_\mathcal{F}} = 0,$
- $\langle \mathcal{P}_z \mathbf{D_z} p, \phi_v\rangle_{\mathcal{H}_P} = 0,$
- $\mathbf{div}_h \mathcal{M}_h v + \mathbf{div}_v w = 0,$

What happens for $\epsilon \to 0$?

Theorem

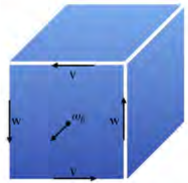
*In the aspect ratio limit, $\epsilon \to 0$,
the solution $(u_{\partial P}^{nh}, p^{nh})$ of the (nonhydrostatic) Euler equations converges
to the solution $(u_{\partial P}^{hyd}, p^{hdy})$ of the hydrostatic Euler equations.*

Theorem

In the aspect ratio limit, $\epsilon \to 0$,
the solution $(u^{nh}_{\partial P}, p^{nh})$ of the (nonhydrostatic) Euler equations converges
to the solution $(u^{hyd}_{\partial P}, p^{hdy})$ of the hydrostatic Euler equations.

Proof

- Consider equation for the difference $\delta u := u^{nh}_{\partial P} - u^{hyd}_{\partial P}$.
- Analyze difference of nonlinear terms
  - $\omega_{\partial \hat{P}} \star u_{\partial P}|^{nh}_{h} - \omega_{\partial \hat{P}} \star u_{\partial P}{}^{hyd}_{h}$
  - $w^{nh}\mathcal{P}_h \mathbf{curl}_h u^{nh} - w^{hyd}\mathcal{P}_h \mathbf{D_z} u^{nh}$
    $\sim \mathbf{curl}_h u^{nh} - \mathbf{D_z} u^{nh}$
- Scalar product of difference equation with δu and energy estimate
- $\to$ Horizontal $\mathbf{curl}_h$ is crucial for estimate



Theorem is discrete version of PDE result by J. Li and E.S. Titi (2019)

Let $\mathcal{G} = \Delta$ be a triangular grid.

$$\textbf{Velocity}: \left\langle \frac{d}{dt} M_h v, \phi \right\rangle + \left\langle \hat{\mathcal{P}}^T [(f+\omega)\hat{\mathcal{P}} v], \phi \right\rangle + \left\langle \mathcal{P}^T \mathcal{Q}(w \mathbf{D_z} \mathcal{P} v), \phi \right\rangle + \left\langle \mathcal{M} \mathbf{grad}[\frac{|\mathcal{P} v|^2_{\mathbb{R}^3}}{2}], \phi \right\rangle + \left\langle \mathcal{P}^T \mathcal{P} \mathbf{grad}(g\eta + p_{hyd}), \phi \right\rangle - \left\langle Lv, \phi \right\rangle = \left\langle \mathcal{F}_v, \phi \right\rangle$$

$$\textbf{Incompress.}: \mathbf{div}_h \mathcal{M}_h v + \mathbf{D_z} w = 0$$

$$\textbf{Free Surface}: \left\langle \frac{\partial \eta}{\partial t}, \psi \right\rangle + \left\langle \mathbf{div}[\sum_{k=0}^{k=N_{top}} \mathcal{P}^T (\Delta z_k \mathcal{P} v_k)], \psi \right\rangle = 0$$

$$\textbf{Tracer}: \left\langle \frac{\partial C}{\partial t}, \psi \right\rangle - \left\langle \mathbf{div}^{up} \mathcal{P}^T (C \mathcal{P} v), \psi \right\rangle + \left\langle LC, \psi \right\rangle = \left\langle \mathcal{F}_C, \psi \right\rangle$$

P. K. Formulation of an Unstructured Grid Model for Global Ocean Dynamics (J. Comp. Phys. 339 (2017))

Theorem

- *Let a vertical mixing scheme of PP-type be active.*

Then the semi-discrete hydrostatic Boussinesq ICON-O equations with a free surface have a unique solution, provided the forcing is sufficiently "nice".

Corollary

The same statement applies if the mesoscale eddy parametrization of Gent-McWilliams-Redi is included and discretized by structure-preserving numerics [a].

[a] *P. K. A structure-preserving discretization of ocean parametrizations on unstructured grids (Ocean Modell. (2018))*

Now allow the density to vary and to compress

Compressible Euler: Momentum vs Velocity

Momentum:

$$\partial_t \rho + div(\rho \mathbf{v}) = 0,$$

$$\partial_t(\rho \mathbf{v}) + \rho\, curl\, \mathbf{v} \times \mathbf{v} + \rho \nabla\left(\frac{|\mathbf{v}|^2}{2} + \Phi\right) + \mathbf{v} div(\rho \mathbf{v}) + \nabla p = 0,$$

$$\partial(\rho e) + div\big(v(\rho e + p)\big) = 0,$$

Energy: $\rho e := \frac{|\mathbf{v}|^2}{2} + c_V T + \rho \Phi,$ EOS: $p = \rho R T.$

Velocity:

$$\partial_t \rho + div(\rho \mathbf{v}) = 0,$$

$$\partial_t \mathbf{v} + curl\, \mathbf{v} \times \mathbf{v} + \nabla\left(\frac{|\mathbf{v}|^2}{2} + \Phi\right) + \frac{\nabla p}{\rho} = 0,$$

$$\partial(\rho e) + div\big(v(\rho e + p)\big) = 0,$$

We use velocity form in analogy with ICON-A.
Similar results for momentum form.

Discrete Compressible Euler

- $\langle \frac{d}{dt}\mathcal{M}u_{\partial P}, \phi\rangle_{\mathcal{H}_{\partial P}} + \langle \omega_{\partial \hat{P}} \star u_{\partial P} + \mathcal{M}\mathbf{grad}(\frac{|\mathcal{P}u_{\partial P}|^2_{\mathbb{R}^3}}{2}), \phi\rangle_{\mathcal{H}_{\partial P}}$
  $+ \langle \mathcal{P}^T(\frac{1}{\rho}\mathcal{P}\mathbf{grad}p)), \phi\rangle_{\mathcal{H}_{\partial P}} = \langle \mathcal{M}\mathbf{grad}\Phi, \phi\rangle_{\mathcal{H}_{\partial P}},$
- $\langle \partial_t \rho + \mathbf{div}^{up}(\mathcal{P}^T(\rho\mathcal{P}u_{\partial P}), \psi\rangle_{\mathcal{H}_P} = 0,$
- $\langle \partial_t(\rho\theta) + \mathbf{div}(\mathcal{P}^T(\rho\theta\mathcal{P}u_{\partial P}), \psi\rangle_{\mathcal{H}_P} = 0,$

Theorem (Well-Posedness of Compressible Euler Equations)

Let a time interval $[0, T]$ *and initial conditions*

- $u_{\partial P}(t = 0) = u_0$, *and* $\theta(t = 0) = \theta_0$
- $\rho(t = 0) = \rho_0$ *with* $\rho_0 \geq c > 0$ *be given.*

Then there exist for $t \in [0, T]$ *a unique solution* $u_{\partial P}(t)$ *of the discrete compressible Euler equations.*

We need to assume upwind advection for ρ *to avoid vacuum.*

Theorem

Solution $u_{\partial P}(t)$ of discrete compressible Euler equations satisfies

- **Energy Conservation:** *The sum of kinetic, potential and internal energy is conserved*

$$\frac{d}{dt}(E^{kin} + E^{pot} + E^{int})(t) = 0, \quad (E^{int} := c_V \rho\theta)$$

- **Helicity Conservation:** *The helicity is conserved*

$$d_t \mathbf{H} = 0.$$

Isentropic Euler Equations with Pressure Equation

$$\partial_t p + \mathbf{v} \cdot \nabla p + \gamma p \, div(\mathbf{v}) = 0, \qquad (\gamma := \frac{c_v}{c_p})$$

$$\partial_t \mathbf{v} + curl \, \mathbf{v} \times \mathbf{v} + \nabla\Big(\frac{|\mathbf{v}|^2}{2} + \Phi\Big) + \frac{\nabla p}{\rho} = 0.$$

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Discrete Isentropic Euler Equations

- $\big\langle \partial_t p + \mathbf{div}[\mathcal{P}^T(\gamma p)\mathcal{P} u_{\partial P}], 1\big\rangle_{\mathcal{H}_P} + \big\langle \gamma' p, \mathbf{div}\mathcal{M} u_{\partial P}\big\rangle_{\mathcal{H}_P} = 0$
- $\big\langle \frac{d}{dt}\mathcal{M} u_{\partial P}, \phi\big\rangle_{\mathcal{H}_{\partial P}} + \big\langle \omega_{\partial\hat{P}} \star u_{\partial P} + \mathcal{M}\mathbf{grad}\big(\frac{|\mathcal{P} u_{\partial P}|^2_{\mathbb{R}^3}}{2}\big), \phi\big\rangle_{\mathcal{H}_{\partial P}}$
  $+ \big\langle \mathcal{P}^T(\frac{1}{\rho}\mathcal{P}\mathbf{grad} p)\big), \phi\big\rangle_{\mathcal{H}_{\partial P}} = \big\langle \mathcal{M}\mathbf{grad}\Phi, \phi\big\rangle_{\mathcal{H}_{\partial P}},$

Theorem

Compressible-Incompressible

- $(u^{\epsilon}_{\partial P}, p^{\epsilon})$ *solution of compressible Euler eq.*
- $(u_{\partial P}, p)$ *solution of incompressible Euler eq.*
- *well-prepared initial conditions:*
  $\mathbf{div} u^{\epsilon}_{\partial P}(t=0) = \mathcal{O}(\epsilon), \quad p^{\epsilon}(t=0) = p(t=0) + \mathcal{O}(1)$

Then solution of compressible equations $({u_{\partial P}}^{\epsilon}, p^{\epsilon})$ *can be written as*

$$u^{\epsilon}_{\partial P} = \underbrace{u + U}_{\text{slow part}} + \tilde{U} + \mathcal{O}(\epsilon), \qquad p^{\epsilon} = \underbrace{p + P}_{\text{slow part}} + \tilde{P} + \mathcal{O}(\epsilon),$$

where

- (U, P) *solution to linearized incompressible Euler*
- $(\tilde{U}, \tilde{P})$ *solution to equations of linear acoustics*
  $\partial_{tt} P' = \Delta_{\mathcal{M}} P', \quad \mathbf{curl} U' = 0 \quad (\Delta_{\mathcal{M}} u := \mathbf{div} \mathcal{M} \mathbf{grad} u)$.

(Proof by analysis of multiscale expansion w.r.t. ϵ)

Discrete version of classical PDE-results from Klainermann-Majda, Kreiss, Schochet. . . .

I will discuss the following topics

- i Incompressible Dynamics *($\sim$ ocean)*
- ii Compressible Dynamics *($\sim$ atmosphere)*
- iii Singular Limits *(relation between different equations)*
- iv Lessons learned

Observation

- Discrete differential operators & reconstructions mesh-unaware

Consequence: Mesh-Independence

- Results valid for:
  triangular △, hexagonal ⬡ and rectangular □ cells
- Results valid for mixed grids □△⬡△△□□ or Delauny-Voronoi polygons.

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Case of rectangular grids □

- Discrete differential operators become classical finite differences
- Reconstructions become familiar averages
- **Nonhydrostatic:** MAC method for Navier-Stokes (Harlow-Welch,1965)
- **Hydrostatic Boussinesq:** same velocity eq. as *NEMO*
  - Nonlinearity conserves 3D-Energy and in 2D energy & enstropy
  - This is again also valid for triangular and hexagonal meshes

2D Incompressible Euler: $\partial_t \omega + v \cdot \nabla \omega = 0$

- $\partial_t \triangle \psi + \mathcal{J}(\psi, \triangle \psi) = 0$
- stream function $v := \nabla^{\perp} \psi$, $\omega = \triangle \psi$ Jacobian $\mathcal{J}$

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Arakawa's Jacobian $\mathcal{J}$ conserves energy & enstrophy on quads

$$\mathcal{J}(\psi, \triangle\psi) = \tfrac{1}{3}\mathcal{J}_1(\psi, \triangle\psi) + \tfrac{1}{3}\mathcal{J}_2(\psi, \triangle\psi) + \tfrac{1}{3}\mathcal{J}_3(\psi, \triangle\psi),$$

$$\mathcal{J}_1(p, q) := \delta_{2x} p \delta_{2y} q - \delta_{2x} q \delta_{2y} p,$$
$$\mathcal{J}_2(p, q) := \delta_{2x}\big(p \delta_{2y} q\big) - \delta_{2y}\big(p \delta_{2x} q\big),$$
$$\mathcal{J}_3(p, q) := \delta_{2y}\big(q \delta_{2x} p\big) - \delta_{2x}\big(q \delta_{2y} p\big).$$

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Arakawa's Jacobian and $\hat{\mathcal{P}}^\dagger(\omega \hat{\mathcal{P}} v)$

$\mathcal{J}(\psi, \triangle\psi) = \mathcal{K}(\psi, \triangle\psi)$
with $\mathcal{K}(\psi, \triangle\psi) := \mathbf{curl}\hat{\mathcal{P}}^\dagger(\triangle\psi \hat{\mathcal{P}} \mathbf{grad}_\tau \psi)$
$\to$ This suggests $\mathcal{K}$ as a generalization of Arakawa's Jacobian to general grids.

C-staggering

Preference of vector invariant nonlinearity: $curl v \times v + \frac{|v|^2}{2}$

First Way to Instability: Specification of Kinetic Energy $\frac{|v|^2}{2}$?

- C-grid models struggle[a] with kinetic energy formulation $|\vec{v}|^2$
- Orthogonal vs non-orthogonal grids
- **Plancherels theorem:** sum of squared components gives vector lengh if and only if components are from orthonormal basis.
- **Rectangular=Orthogonal** : sum of squared components $|\vec{v}|^2 \sim \sum_{e \in \partial \square} |v_e|^2$ is justified
- **Unstructured=Non-orthogonal**: need to rely on square of reconstructed vector $|\vec{v}|^2 \sim |\mathcal{P} v|^2$
  $\rightarrow$ This implies a mass matrix $\mathcal{M}$
- Using sum of squares on unstructured grids creates energy source/sink

[a] *ICON-A: Zängl, QJRMS, 2017, MPAS-A: Skamarock-Klemp, MWR, 2012*

Second Way to Instability: Exterior Product $\omega \times v$

- Mixture of vector-invariant and advective form of nonlinearity *(partly vector invariant, partly advective)*
- Prohibits cancelation of fluxes
- Ambiguous nonlinearity impedes energetic consistency and other conservation properties
- Lack of energetic consistency degrades models stability properties

Time Stepping

Fully discrete conservation laws presented here demand implicit time stepping.

Not Negotiable: Algorithmic Essentials

- Clean kinetic energy definition E^{kin}:
  - **Non-orthogonal grids:** reconstruction-based mandatory
  - **Orthogonal grids:** sum of squares or by reconstruction.
- 3D-vector-invariant form of nonlinearity

Not Negotiable: Algorithmic Essentials

- Clean kinetic energy definition E^{kin}:
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- 3D-vector-invariant form of nonlinearity

Negotiable: Algorithmic Degrees of Freedom

- Reconstructions: different reconstructions can be used
- Vertical coordinates
  (we know how to do this)
- Lumping mas matrix in time derivative: short-cut to inverse $\mathcal{M}^{-1}$
- Higher-Order, upwind-biased reconstructions
- Flux limiters
- Time stepping: alternative time steppings can be used
  (implicit used here for theoretical beauty)

The Discrete Hierarchy of Atmosphere-Ocean Equations