Applying Implicit Energetics Ideas to Simulate Full-Water-Column Ocean Mixing

Robert Hallberg[1,2] and Brandon Reichl[1]
[1]NOAA / GFDL and [2]Princeton University

Climate Change will alter Ocean Mixing

- Mixing stratified fluid takes energy
  - Energy sources for diapycnal mixing will change in an evolving climate, including the external and internal tides
  - Evolving stratification changes turbulent energetics
- Fixed diapycnal diffusivities will be replaced in GFDL's new MOM6-based OM5/CM5/ESM5 hierarchy of global ocean-climate models by parameterizations of energy inputs and an Integrated Implicit Energetics Approach

Diapycnal Mixing, Turbulent Kinetic Energy and the Osborn Relationship

A simplified, dominant balance for the turbulent kinetic energy budget allows (hard to observe) turbulent vertical buoyancy fluxes to be related to the (more easily inferred) dissipation of turbulent kinetic energy. (Osborn, *JPO* 1980)

A typical turbulent kinetic energy (TKE or E) balance equation:

$$E \equiv \tfrac{1}{2}\left(\overline{u'^2 + v'^2 + w'^2}\right)$$

$$\frac{DE}{Dt} = \nabla \cdot \mathbf{F_Q} - \overline{\mathbf{u}'w'}\frac{\partial \bar{\mathbf{u}}}{\partial z} + \overline{w'b'} - \nu_{mol}\overline{\nabla \mathbf{u}' \cdot \nabla \mathbf{u}'}$$

$$\approx \frac{\partial}{\partial z}\left(\nu_E \frac{\partial E}{\partial z}\right) + \nu_u \left\| \frac{\partial \bar{\mathbf{u}}}{\partial z} \right\|^2 + Src_{IGW} - \kappa N^2 - \varepsilon$$

$$N^2 \equiv -\frac{g}{\rho}\frac{\partial \rho}{\partial z}$$

$\kappa \equiv$ Turbulent diffusivity of heat & salt

In the interior ocean, the local balance often works pretty well:

$$0 \approx Src_{Shear} + Src_{IGW} - \kappa N^2 - \varepsilon$$

$$P = \kappa N^2 + \varepsilon$$

$\varepsilon \equiv$ Dissipation of TKE

$P \equiv$ Production of TKE

The flux Richardson number $\left(R_f \equiv \kappa N^2 / P\right)$ is typically less than a critical value of ~0.15.

$$\kappa N^2 / R_f = \kappa N^2 + \varepsilon$$

$$\boxed{\kappa = \frac{\Gamma \varepsilon}{N^2}}$$

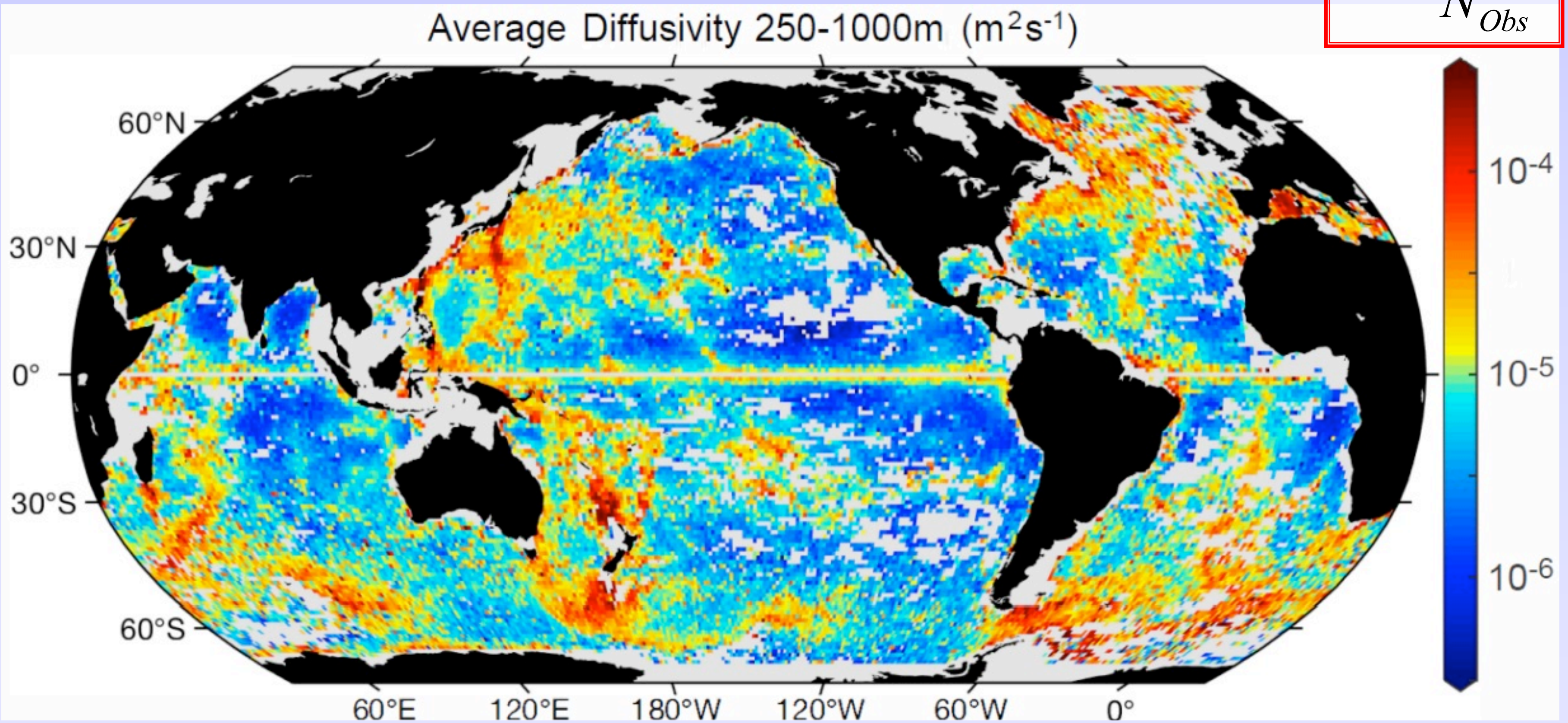
$$\Gamma \equiv \frac{R_f}{1 - R_f} \lesssim 0.2$$

The buoyancy frequency and dissipation can both be determined observationally.

The Osborn Relationship can be used to estimate diffusivities from observed small-scale dissipation

$$\kappa = \frac{\Gamma \varepsilon_{Obs}}{N_{Obs}^2}$$

Estimated Ocean Diffusivity Based on ARGO Observations

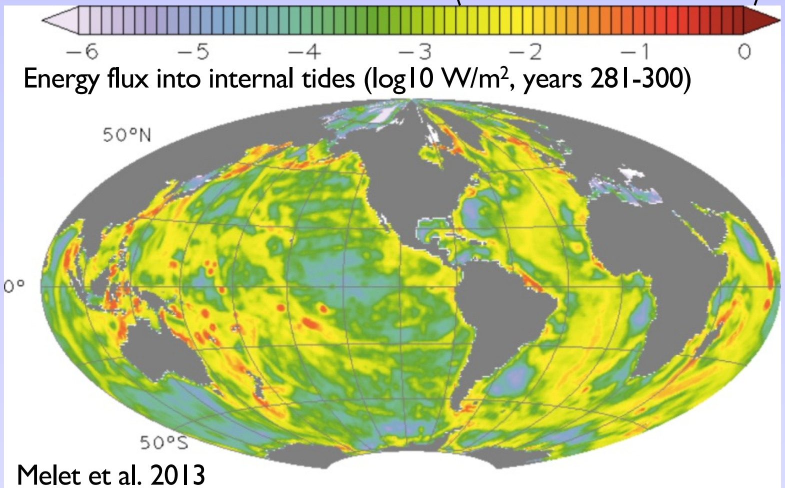


Bindoff et al. (2019) (IPCC SROCC, Ch. 5)

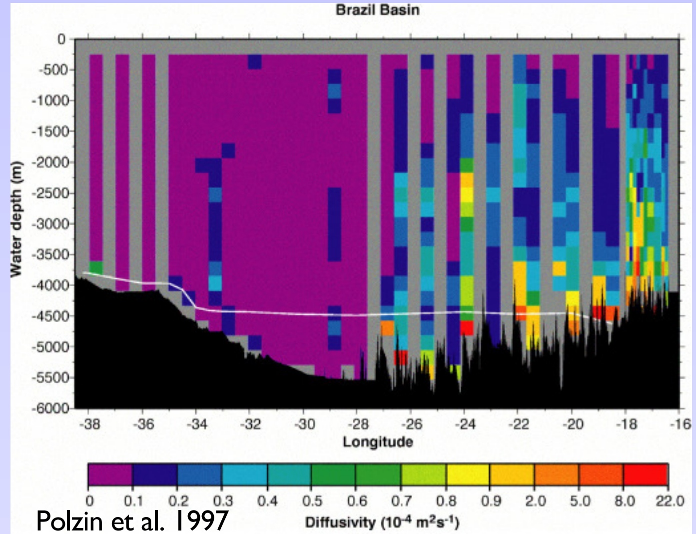
Updated from Whalen, Talley and MacKinnon, GRL 2012

Parameterizing breaking high-mode internal tides

$$E(x,y)=\tfrac{1}{2}\rho N_{Bot} k h^2 \left\langle U^2_{\text{bot,model}} + u^2_{\text{Tide}} \right\rangle$$



map of internal tide generation from model's evolving U_{bot} & N_{bot})



vertical structure based on wave-wave interaction physics (St Laurent et al 2002 or Polzin 2009)

Osborn relationship:

$$\kappa = \frac{\Gamma \varepsilon}{N^2}$$

$$\Gamma \equiv \frac{R_f}{1-R_f} \leq\sim 0.2$$

dissipation rate (related to diffusivity through mixing efficiency)

$$\varepsilon = \frac{qE(x,y)F(z)}{\rho}$$

% of energy that dissipates locally

St Laurent et al 02; Polzin 09; Melet et al 13; Melet et al 15; Internal Wave Mixing CPT

Mixing in a stratified water columns takes energy.

- Osborn Relationship:

$$\kappa = \frac{\Gamma \varepsilon}{N^2}$$

$$\Gamma \equiv \frac{R_f}{1 - R_f} \leq\sim 0.2$$

Turbulent Kinetic Energy supplies the local potential energy change due to the local buoyancy flux.

- New Implicit Energetics approach:

Turbulent Kinetic Energy supplies the potential energy changes throughout the entire water column integrated over a timestep due to the local diffusivity.

These are the same in the limit where $\Delta t \rightarrow 0$.

Potential Energy of a Hydrostatic Column (Exact)

$$PE = \int_{-D}^{\eta} \rho g \left(z' + D \right) dz$$

$$= -\int_{p_D}^{0} \left(z + D \right) dp$$

$$= \int_{0}^{p_D} \int_{p}^{p_D} R \frac{dp'}{g} dp$$

$$= \int_{0}^{p_D} R \frac{1}{g} p dp + \left[p \int_{p}^{p_D} R \frac{1}{g} dp \right]_{p=0}^{p_D}$$

$$PE = \int_{0}^{p_D} R \frac{1}{g} p dp$$

- Linear in specific volume
- Fixed bounds of integration in pressure

Specific Volume: $R \equiv \frac{1}{\rho}$

Hydrostatic Balance:

$$\frac{dp}{dz} = -g\rho$$

Height above the bottom (D) as a function of pressure:

$$z(p) = -D + \int_{p_D}^{p} \frac{dz}{dp} dp'$$

$$= -D - \int_{p_D}^{p} \frac{R}{g} dp'$$

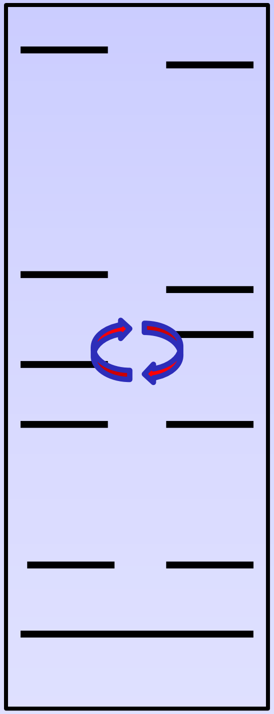
Integration by parts:

$$\int u dv = uv - \int v du$$

PE Change from Mixing & Conversion to TKE

Change in potential energy due to diffusivity κ_k at interface k:

$$\frac{d\dot{PE}}{d\kappa_k} = \int_0^{p_D} \frac{d\dot{R}}{d\kappa_k} \frac{1}{g} p\,dp \qquad \text{Specific Volume: } R \equiv \frac{1}{\rho}$$



A fraction n*≈0.07? of released PE is available to drive more mixing, but energy released by contraction of the column radiates as gravity waves:

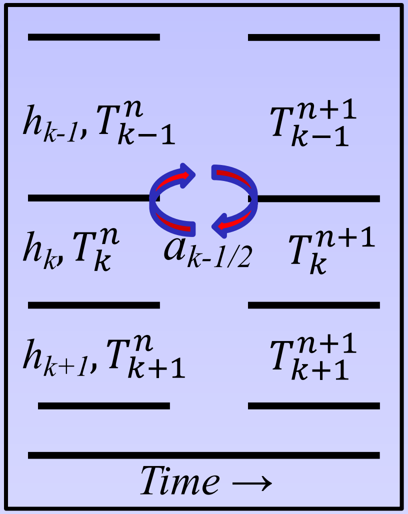
$$\frac{d\dot{TKE}}{d\kappa_k} = -\left[\int_0^{p_D} \frac{d\dot{R}}{d\kappa_k} \frac{1}{g} p\,dp - \min\left(0, \frac{p_D}{g} \int_0^{p_D} \frac{d\dot{R}}{d\kappa_k} dp\right)\right]\begin{pmatrix} 1 & \frac{d\dot{TKE}}{d\kappa} < 0 \\ n^* & \frac{d\dot{TKE}}{d\kappa} > 0 \end{pmatrix}$$

Mixing is done for conservative temperature and salinity:

$$\int_0^{p_D} \frac{d\dot{R}}{d\kappa_k} \frac{1}{g} p\,dp = \int_0^{p_D} \left(\frac{\partial R}{\partial \theta} \frac{d\dot{\theta}}{d\kappa_k} + \frac{\partial R}{\partial S} \frac{d\dot{S}}{d\kappa_k} \right) \frac{1}{g} p\,dp$$

The tridiagonal equations for the total implicit evolution of θ and S profiles can be differentiated with κ_k and integrated over all the layers above & below without having to re-traverse the water column.

Tridiagonal Solvers for Implicit Mixing



Implicit finite volume tracer diffusion equation:

$$h_k \frac{T_k^{n+1} - T_k^n}{\Delta t} = \kappa_{k-1/2} \frac{T_{k-1}^{n+1} - T_k^{n+1}}{\Delta z_{k-1/2}} - \kappa_{k+1/2} \frac{T_k^{n+1} - T_{k+1}^{n+1}}{\Delta z_{k+1/2}}$$

$$a_{k-1/2} = \frac{\kappa_{k-1/2}\Delta t}{\Delta z_{k-1/2}} \qquad a_{1/2} = a_{K+1/2} = 0$$

$$\left(h_k + a_{k+1/2} + a_{k-1/2}\right)T_k^{n+1} = h_k T_k^n + a_{k-1/2}T_{k-1}^{n+1} + a_{k+1/2}T_{k+1}^{n+1}$$

How NOT to write the solver:

$$b_k = \left(h_k + a_{k-1/2} + a_{k+1/2}\right)$$

$$q_k = \frac{a_{k+1/2}}{b_k - a_{k-1/2}q_{k-1}}$$ ***Dangerous subtraction!***

$$\tilde{T}_k = \frac{h_k T_k^n + a_{k-1/2}\tilde{T}_{k-1}}{b_k - a_{k-1/2}q_{k-1}}$$

$$\ldots$$

$$T_k^{n+1} = \tilde{T}_k + q_k T_{k+1}^{n+1}$$

$$q_0 = q_{K+1/2} = 0$$

"Thomas algorithm", Thomas (*1949*)

The robust way to write the solver:

$$Q_k \equiv 1 - q_k$$

$$q_k = \frac{a_{k+1/2}}{h_k + a_{k+1/2} + a_{k-1/2} - q_{k-1}a_{k-1/2}}$$

$$= \frac{a_{k+1/2}}{h_k + Q_{k-1}a_{k-1/2} + a_{k+1/2}}$$ ***No subtraction!***

$$Q_k = \frac{h_k + Q_{k-1}a_{k-1/2}}{h_k + Q_{k-1}a_{k-1/2} + a_{k+1/2}}$$

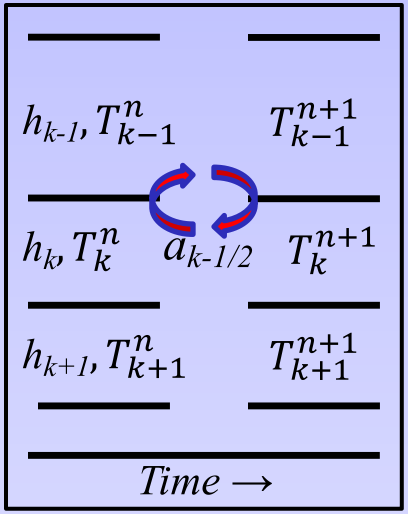
$$\tilde{T}_k = \frac{h_k T_k^n + a_{k-1/2}\tilde{T}_{k-1}}{h_k + Q_{k-1}a_{k-1/2} + a_{k+1/2}}$$

$$\ldots$$

$$T_k^{n+1} = \tilde{T}_k + q_k T_{k+1}^{n+1}$$

Schopf & Loughe (*MWR*, 1995)

Three Valid Tridiagonal Solvers for Implicit Mixing



$$h_k \frac{T_k^{n+1} - T_k^n}{\Delta t} = \kappa_{k-1/2}\frac{T_{k-1}^{n+1} - T_k^{n+1}}{\Delta z_{k-1/2}} - \kappa_{k+1/2}\frac{T_k^{n+1} - T_{k+1}^{n+1}}{\Delta z_{k+1/2}}$$

$$a_{k-1/2} = \frac{\kappa_{k-1/2}\Delta t}{\Delta z_{k-1/2}} \qquad a_{1/2} = a_{K+1/2} = 0$$

$$\left(h_k + a_{k+1/2} + a_{k-1/2}\right)T_k^{n+1} = h_k T_k^n + a_{k-1/2}T_{k-1}^{n+1} + a_{k+1/2}T_{k+1}^{n+1}$$

Downward First:

$$Q_{1/2}^{\Downarrow} a_{1/2} = 0 \qquad q_{K+1/2}^{\Downarrow} = 0$$

$$\ldots$$

$$h_k^{\Downarrow} = h_k + Q_{k-1/2}^{\Downarrow} a_{k-1/2}$$

$$q_{k+1/2}^{\Downarrow} = \frac{a_{k+1/2}}{h_k^{\Downarrow} + a_{k+1/2}}$$

$$Q_{k+1/2}^{\Downarrow} = \frac{h_k^{\Downarrow}}{h_k^{\Downarrow} + a_{k+1/2}}$$

$$\tilde{T}_k^{\Downarrow} = \frac{h_k T_k^n + a_{k-1/2}\tilde{T}_{k-1}^{\Downarrow}}{h_k^{\Downarrow} + a_{k+1/2}}$$

$$\ldots$$

$$T_k^{n+1} = \tilde{T}_k^{\Downarrow} + q_{k+1/2}^{\Downarrow} T_{k+1}^{n+1}$$

Upward First:

$$Q_{K+1/2}^{\Uparrow} a_{K+1/2} = 0 \qquad q_{1/2}^{\Uparrow} = 0$$

$$\ldots$$

$$h_k^{\Uparrow} = h_k + Q_{k+1/2}^{\Uparrow} a_{k+1/2}$$

$$q_{k-1/2}^{\Uparrow} = \frac{a_{k-1/2}}{h_k^{\Uparrow} + a_{k-1/2}}$$

$$Q_{k-1/2}^{\Uparrow} = \frac{h_k^{\Uparrow}}{h_k^{\Uparrow} + a_{k-1/2}}$$

$$\tilde{T}_k^{\Uparrow} = \frac{h_k T_k^n + a_{k+1/2}\tilde{T}_{k+1}^{\Uparrow}}{h_k^{\Uparrow} + a_{k-1/2}}$$

$$\ldots$$

$$T_k^{n+1} = \tilde{T}_k^{\Uparrow} + q_{k-1/2}^{\Uparrow} T_{k-1}^{n+1}$$

Inward First:

$$\ldots$$

$$\tilde{Q}_{k+1/2}^{\Updownarrow} = \frac{1}{1 - q_{k+1/2}^{\Downarrow} q_{k+1/2}^{\Uparrow}} = \frac{\left(h_k^{\Downarrow} + a_{k+1/2}\right)\left(h_{k+1}^{\Uparrow} + a_{k+1/2}\right)}{h_k^{\Downarrow} h_{k+1}^{\Uparrow} + a_{k+1/2}\left(h_k^{\Downarrow} + h_{k+1}^{\Uparrow}\right)}$$

$$T_k^{n+1} = \tilde{Q}_{k+1/2}^{\Updownarrow}\left(\tilde{T}_k^{\Downarrow} + q_{k+1/2}^{\Uparrow}\tilde{T}_{k+1}^{\Uparrow}\right)$$

$$T_{k+1}^{n+1} = \tilde{Q}_{k+1/2}^{\Updownarrow}\left(q_{k+1/2}^{\Downarrow}\tilde{T}_k^{\Downarrow} + \tilde{T}_{k+1}^{\Uparrow}\right)$$

$$\ldots$$

Vertically Integrated Potential Energy Change due to Diffusion

$$\frac{d\dot{P}E}{d\kappa_{k-1/2}} = \int_0^{p_D} \left(\frac{\partial R}{\partial \theta} \frac{d\dot{\theta}}{d\kappa_{k-1/2}} + \frac{\partial R}{\partial S} \frac{d\dot{S}}{d\kappa_{k-1/2}} \right) p \frac{1}{g} dp$$

$$\Delta_{\Delta\kappa_{k+1/2}} PE = \frac{1}{\Delta t} \sum_{j=1}^{K} \left(\frac{\partial R}{\partial \theta}_j \Delta_{\Delta\kappa_{k+1/2}} \theta_j + \frac{\partial R}{\partial S}_j \Delta_{\Delta\kappa_{k+1/2}} S_j \right) \frac{1}{g} \bar{p}_j \Delta p_j$$

$$\Delta_{\Delta\kappa_{k+1/2}} PE = \frac{W_{k+1/2}}{A} \frac{\left(\Delta\kappa_{k+1/2}\Delta t / \Delta z_{k+1/2}\right)}{A + B\left(\Delta\kappa_{k+1/2}\Delta t / \Delta z_{k+1/2}\right)}$$

$$A \equiv h_k^{\Downarrow} h_{k+1}^{\Uparrow} + \frac{\kappa_{k+1/2}^{prev}\Delta t}{\Delta z_{k+1/2}} \left(h_k^{\Downarrow} + h_{k+1}^{\Uparrow} \right)$$

$$B \equiv h_k^{\Downarrow} + h_{k+1}^{\Uparrow}$$

$$W_{k+1/2} \equiv \left(h_{k+1}^{\Uparrow} \frac{\partial PE}{\partial \theta}_k^{\Downarrow} - h_k^{\Downarrow} \frac{\partial PE}{\partial \theta}_{k+1}^{\Uparrow} \right) \left(h_k^{\Downarrow} h\tilde{\theta}_{k+1}^{\Uparrow} - h_{k+1}^{\Uparrow} h\tilde{\theta}_k^{\Downarrow} \right) +$$

$$\left(h_{k+1}^{\Uparrow} \frac{\partial PE}{\partial S}_k^{\Downarrow} - h_k^{\Downarrow} \frac{\partial PE}{\partial S}_{k+1}^{\Uparrow} \right) \left(h_k^{\Downarrow} h\tilde{S}_{k+1}^{\Uparrow} - h_{k+1}^{\Uparrow} h\tilde{S}_k^{\Downarrow} \right)$$

$$\frac{\partial PE}{\partial \theta}_k^{\Downarrow} = \frac{1}{g} \bar{p}_k \Delta p_k \frac{\partial R}{\partial \theta_k} + q_{k-1/2}^{\Downarrow} \frac{\partial PE}{\partial \theta}_{k-1}^{\Downarrow}$$

$$\frac{\partial PE}{\partial \theta}_k^{\Uparrow} = \frac{1}{g} \bar{p}_k \Delta p_k \frac{\partial R}{\partial \theta_k} + q_{k+1/2}^{\Uparrow} \frac{\partial PE}{\partial \theta}_{k+1}^{\Uparrow}$$

A properly written tridiagonal solver for the implicit finite volume tracer diffusion equation:

$$a_{k+1/2} \equiv \frac{\kappa_{k+1/2}\Delta t}{\Delta z_{k+1/2}}$$

$$\left(h_k + a_{k+1/2} + a_{k-1/2} \right) T_k^{n+1} = h_k T_k^n + a_{k-1/2} T_{k-1}^{n+1} + a_{k+1/2} T_{k+1}^{n+1}$$

$$h_k^{\Downarrow} = h_k + Q_{k-1/2}^{\Downarrow} a_{k-1/2}$$

$$q_{k+1/2}^{\Downarrow} = \frac{a_{k+1/2}}{h_k^{\Downarrow} + a_{k+1/2}}$$

$$Q_{k+1/2}^{\Downarrow} \equiv 1 - q_{k+1/2}^{\Downarrow} = \frac{h_k^{\Downarrow}}{h_k^{\Downarrow} + a_{k+1/2}}$$

$$\tilde{T}_k^{\Downarrow} = \frac{h_k T_k^n + a_{k-1/2} \tilde{T}_{k-1}^{\Downarrow}}{h_k^{\Downarrow} + a_{k+1/2}}$$

…

$$T_k^{n+1} = \tilde{T}_k^{\Downarrow} + q_{k+1/2}^{\Downarrow} T_{k+1}^{n+1}$$

$$h\tilde{T}_k^{\Downarrow} \equiv h_k T_k^n + a_{k-1/2} \tilde{T}_{k-1}^{\Downarrow} \qquad h_k^{\Downarrow} \equiv h_k + Q_{k-1/2}^{\Downarrow} a_{k-1/2}$$

$$h\tilde{T}_k^{\Uparrow} \equiv h_k T_k^n + a_{k+1/2} \tilde{T}_{k+1}^{\Uparrow} \qquad h_k^{\Uparrow} \equiv h_k + Q_{k+1/2}^{\Uparrow} a_{k+1/2}$$

- The buoyancy frequency does not appear anywhere in these expressions!
- The only approximations here are hydrostatic balance and that the thermal expansion and haline contraction coefficients of a layer don't change much over a timestep.

Fully implicit expression for PE change:

$$\Delta_{\Delta\kappa_{k+1/2}} PE = \frac{W_{k+1/2}}{A}\,\frac{\left(\Delta\kappa_{k+1/2}\Delta t / \Delta z_{k+1/2}\right)}{A + B\left(\Delta\kappa_{k+1/2}\Delta t / \Delta z_{k+1/2}\right)}$$

$$A \equiv h_k^{\Downarrow} h_{k+1}^{\Uparrow} + \frac{\kappa_{k+1/2}^{prev}\Delta t}{\Delta z_{k+1/2}}\left(h_k^{\Downarrow} + h_{k+1}^{\Uparrow}\right)$$

$$B \equiv h_k^{\Downarrow} + h_{k+1}^{\Uparrow}$$

$$W_{k+1/2} \equiv \left(h_{k+1}^{\Uparrow} \left.\frac{\partial PE^{\Downarrow}}{\partial \theta}\right|_k - h_k^{\Downarrow} \left.\frac{\partial PE^{\Uparrow}}{\partial \theta}\right|_{k+1} \right)\left(h_k^{\Downarrow} h \tilde{\theta}_{k+1}^{\Uparrow} - h_{k+1}^{\Uparrow} h \tilde{\theta}_k^{\Downarrow}\right) + \left(h_{k+1}^{\Uparrow} \left.\frac{\partial PE^{\Downarrow}}{\partial S}\right|_k - h_k^{\Downarrow} \left.\frac{\partial PE^{\Uparrow}}{\partial S}\right|_{k+1} \right)\left(h_k^{\Downarrow} h \tilde{S}_{k+1}^{\Uparrow} - h_{k+1}^{\Uparrow} h \tilde{S}_k^{\Downarrow}\right)$$

Solve for $\Delta\kappa_{k+1/2}$ given ΔPE:

$$\Delta\kappa_{k+1/2} = \frac{\Delta z_{k+1/2}}{\Delta t}\,\frac{A^2 \Delta PE_{k+1/2}}{W_{k+1/2} - AB\Delta PE_{k+1/2}} \quad for \quad \Delta PE_{k+1/2} < \frac{W_{k+1/2}}{AB}$$

Simplified to mixing between 2 layers, with a linear eqn of state & 1 state variable…

$$\Delta_{\kappa_{3/2}} PE = \frac{W_{3/2}}{h_1 h_2}\,\frac{\Delta\kappa_{3/2}\Delta t}{h_1 h_2 \Delta z_{3/2} + (h_1 + h_2)\Delta\kappa_{3/2}\Delta t}$$

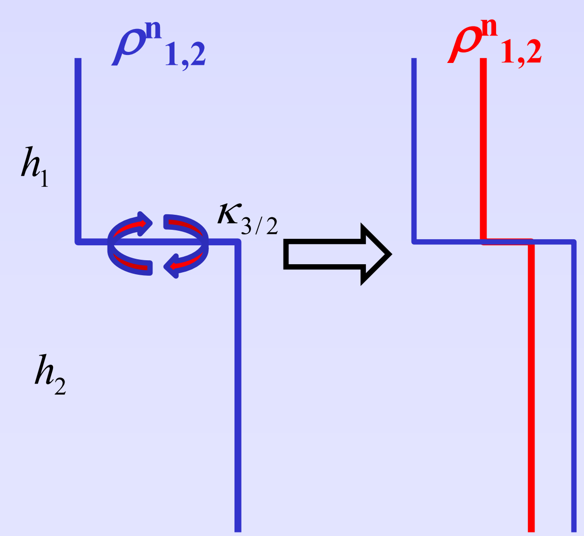
$$W_{3/2} = \rho \frac{\partial R}{\partial \theta} h_1^2 h_2^2 (\bar{p}_1 - \bar{p}_2)(\theta_2^n - \theta_1^n) \qquad \Delta z_{3/2} = \tfrac{1}{2}(h_1 + h_2)$$

$$\approx \tfrac{1}{2} g h_1^2 h_2^2 (h_1 + h_2) \frac{\partial \rho}{\partial \theta} (\theta_1^n - \theta_2^n)$$

$$\Delta_{\kappa_{3/2}} PE = \tfrac{1}{2} g\left(\rho_1^n - \rho_2^n\right) h_1 h_2 \frac{2\Delta\kappa_{3/2}\Delta t}{h_1 h_2 + 2\Delta\kappa_{3/2}\Delta t}$$

$$\Delta_{\kappa_{3/2}=\infty} PE = \tfrac{1}{2} g\left(\rho_1^n - \rho_2^n\right) h_1 h_2$$

$$\Delta_{\kappa_{3/2}\to 0} PE = g\left(\rho_1^n - \rho_2^n\right)\kappa_{3/2}\Delta t$$



Fully implicit expression for PE change:

$$\Delta_{\Delta\kappa_{k+1/2}} PE = \frac{W_{k+1/2}}{A} \frac{\left(\Delta\kappa_{k+1/2}\Delta t / \Delta z_{k+1/2}\right)}{A + B\left(\Delta\kappa_{k+1/2}\Delta t / \Delta z_{k+1/2}\right)}$$

$$A \equiv h_k^{\Downarrow} h_{k+1}^{\Uparrow} + \frac{\kappa_{k+1/2}^{prev}\Delta t}{\Delta z_{k+1/2}}\left(h_k^{\Downarrow} + h_{k+1}^{\Uparrow}\right)$$

$$B \equiv h_k^{\Downarrow} + h_{k+1}^{\Uparrow}$$

$$W_{k+1/2} \equiv \left(h_{k+1}^{\Uparrow} \frac{\partial PE}{\partial \theta}_k^{\Downarrow} - h_k^{\Downarrow} \frac{\partial PE}{\partial \theta}_{k+1}^{\Uparrow} \right)\left(h_k^{\Downarrow} h\tilde{\theta}_{k+1}^{\Uparrow} - h_{k+1}^{\Uparrow} h\tilde{\theta}_k^{\Downarrow} \right) + \left(h_{k+1}^{\Uparrow} \frac{\partial PE}{\partial S}_k^{\Downarrow} - h_k^{\Downarrow} \frac{\partial PE}{\partial S}_{k+1}^{\Uparrow} \right)\left(h_k^{\Downarrow} h\tilde{S}_{k+1}^{\Uparrow} - h_{k+1}^{\Uparrow} h\tilde{S}_k^{\Downarrow} \right)$$

Solve for $\Delta\kappa_{k+1/2}$ given ΔPE:

$$\Delta\kappa_{k+1/2} = \frac{\Delta z_{k+1/2}}{\Delta t} \frac{A^2 \Delta PE_{k+1/2}}{W_{k+1/2} - AB\Delta PE_{k+1/2}} \quad for \quad \Delta PE_{k+1/2} < \frac{W_{k+1/2}}{AB}$$

With weak stratification, energy to mix buoyancy does not bound mixing of other quantities; something else is needed for a reasonable upper bound on diffusivities.

Perhaps a vertical mixing length-scale, *L,* and turbulent velocity, *w*,* that can be combined to give a diffusivity?

$$K_\lambda(z) = C_\lambda w^*(z) L(z)$$

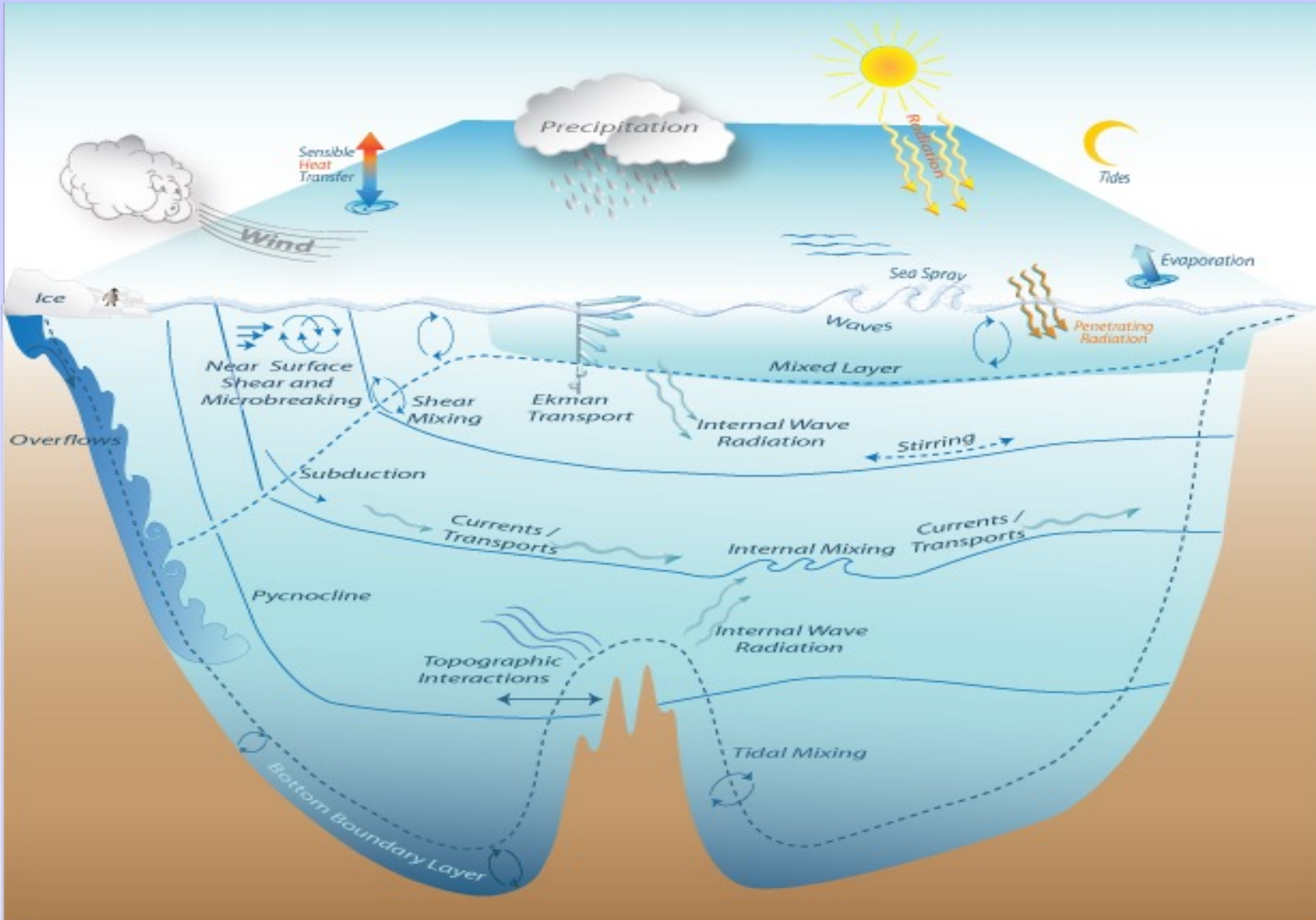
Subject to several considerations:

- $\left|\frac{dL}{dz}\right| \le 1$ (analogous to the law-of-the-wall)
- L goes to 0 at solid boundaries or where stratification suppresses mixing
- A turbulent velocity could itself be related to a turbulent energy flux or stress:

  e.g., if $\frac{\Delta PE_{k+1/2}}{\Delta t} = \Delta z_{k+1/2} \frac{dF_{TKE}}{dz}$, perhaps $w^* \propto \sqrt[3]{F_{TKE}/\rho}$
- The turbulent mixing can auto-diffuse the turbulent velocity and length scales?

The details of the peak diffusivity will depend on the mixing process.

Energetic considerations can be used with the new integrated approach to parameterize many of the mixing processes in the ocean.



An ‘ePBL’ Framework: The energetics concept of a Kraus-Turner-Niiler boundary layer scheme with a KPP-like finite diffusivity

Kraus-Turner-Niiler Bulk Mixed Layer: Integrated energetics is used to determine the boundary layer depth (h) or entrainment rate (w_E).

Mixing | *Mechanical* | *Convection*

$$\frac{h}{2} g' \max\left(w_E, 0\right) + \frac{h}{2} \max\left(B_0, 0\right) = m_* u_*^3 - n_* \frac{h}{2} \min\left(B_0, 0\right)$$

ePBL: Similar integrated energetics concept, but with finite turbulent mixing coefficients (diffusivity & viscosity); H_{bl} is the depth at which the TKE is used up:

Mixing | *Mechanical* | *Convection*

$$\int_{-H_{bl}}^{0} \max\left(N^2 K(z), 0\right) dz = m_* u_*^3 - n_* \int_{-H_{bl}}^{0} \min\left(N^2 K(z), 0\right) dz$$

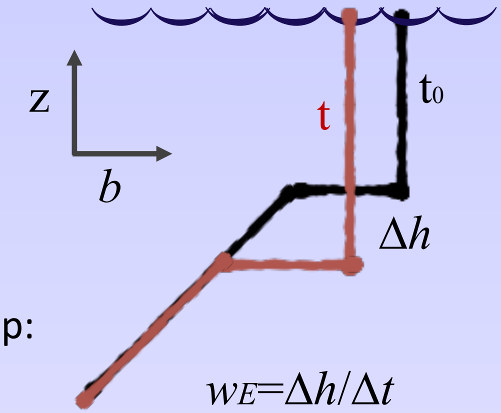
This uses the shorthand: $N^2 K(z) \equiv \frac{1}{\rho_o} \int_0^{K(z)} \left(\int_0^{p_D} \frac{d\dot{R}}{d\kappa_z} \frac{1}{g} p dp \right) d\kappa_z$

Mixing Coefficients: $K_\lambda(z) = C_\lambda w(z) L(z)$

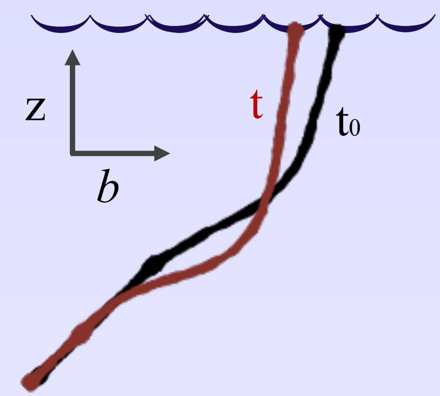
Mixing Length: $L(z) = (z_0 + |z|) max\left[\frac{l_b}{H_{bl}}, \left(\frac{H_{bl} - |z|}{H_{bl}}\right)^{\gamma}\right]$

Velocity Scale: $w(z) = C_{w_*} \left(\int_z^0 \overline{w'b'}\, dz \right)^{1/3} + C_{u_*} u_* \left[1 - a \cdot min\left(1, \frac{|z|}{H_{bl}}\right)\right]$

Kraus-Turner-Niiler



ePBL



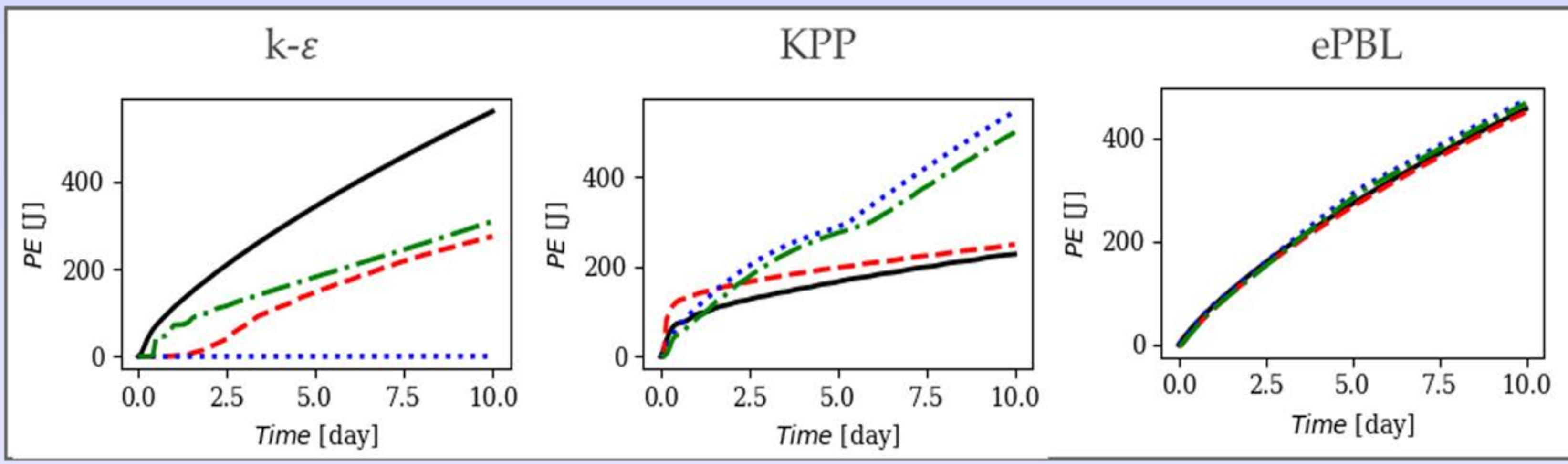
See Reichl and Hallberg (2018, *Ocean Modelling*) for full details.
See also Li, Reichl, et al., (2019, *JAMES*) for an Intercomparison.

Energetics Planetary Boundary Layer (ePBL) numerics: Robust model solution to grid resolution and time-step

A simple wind-driven test case of mixing into a stratified water column demonstrates the very weak dependence on vertical resolution and timestep arising from the integrated energetics approach.

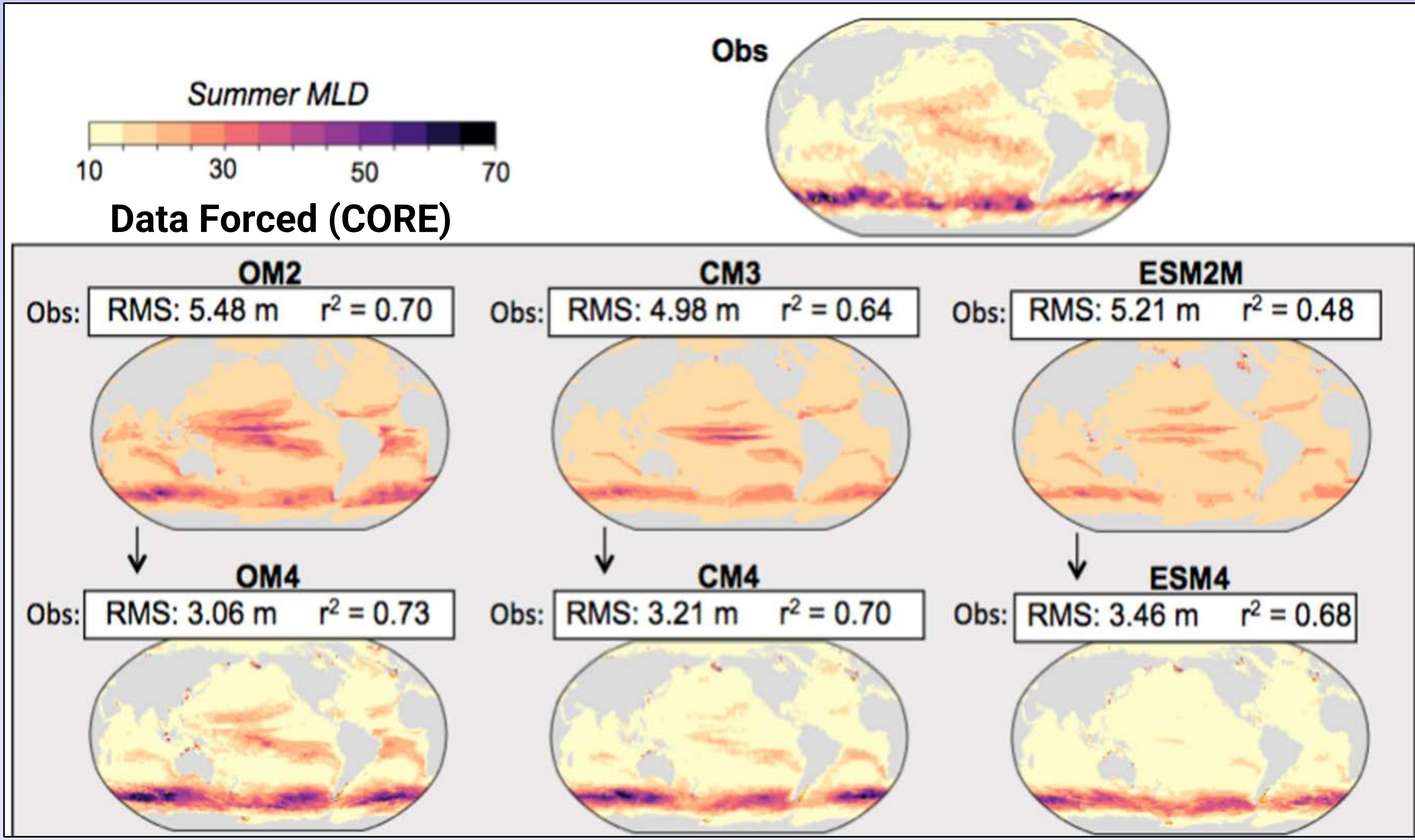
1-d Wind-driven Simulations $\tau = 0.25\ \mathrm{N/m^2}$

$f = 2\Omega \sin(60)\ \mathrm{s^{-1}}$



ΔZ=1 m, ΔT30 s ΔZ=20 m, ΔT30 s ΔZ=1 m, ΔT7200 s ΔZ=20 m, ΔT7200 s

Results: ePBL (here with modifications for Langmuir Turbulence) contributes to improved Mixed Layer Depths in climate models

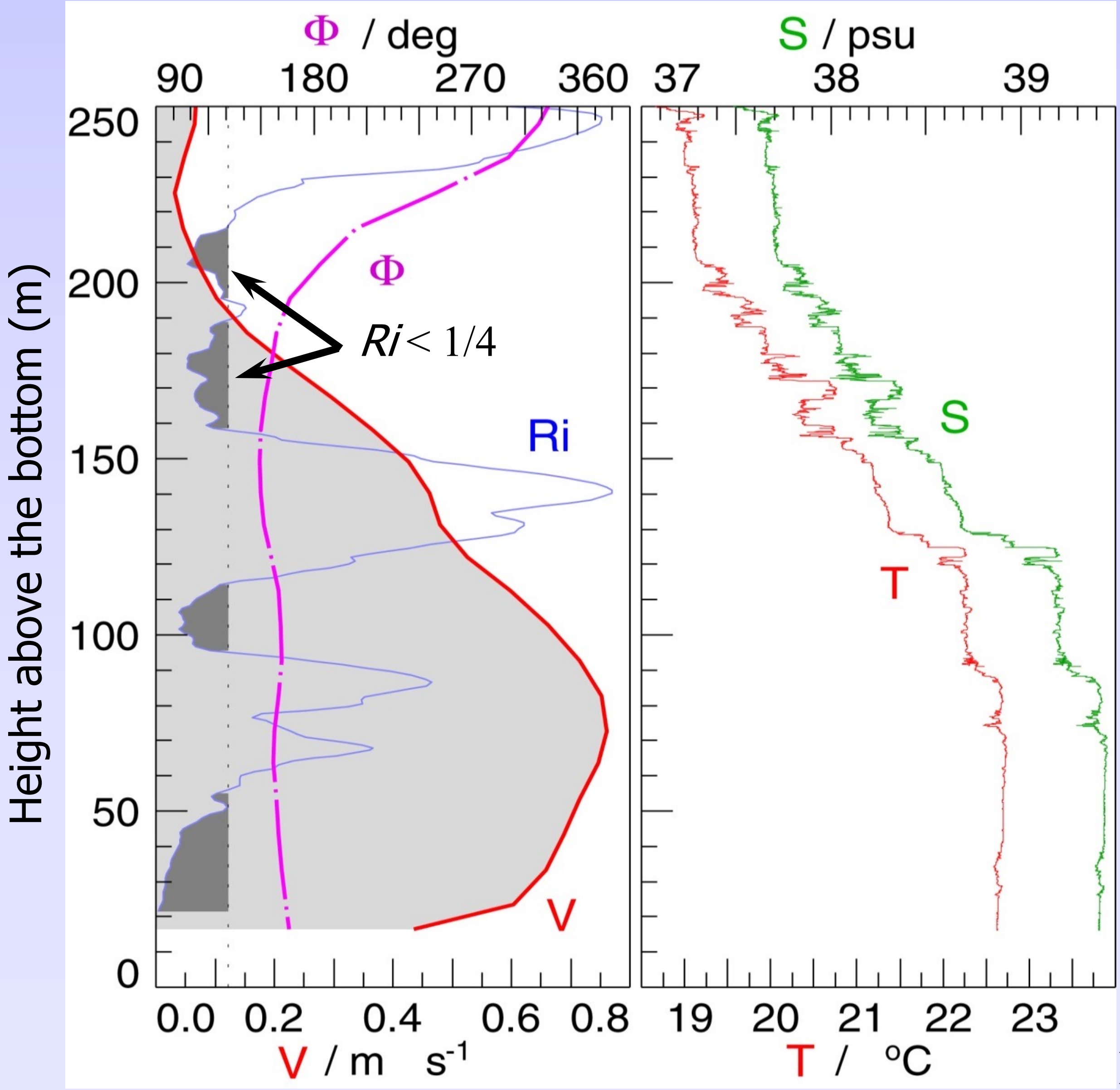


Adcroft et al. (2019, *JAMES*) Held et al. (2019, *JAMES*) Dunne et al. (2020, *JAMES*)

A Bottom Boundary Layer version of ePBL is now available in MOM6.

Shear-driven Mixing in a (Bottom) Boundary Layer

Observed profiles from the Red Sea outflow plume



$$\|Sh\|^2 = \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2$$

$$N^2 = -\frac{g}{\rho}\frac{\partial \rho}{\partial z}$$

$$Ri = \frac{N^2}{\|Sh\|^2}$$

Actively mixing Interfacial Layer

Shear Ri# Param. Appropriate Here.

Well-mixed Bottom Boundary Layer

Courtesy H. Peters, see Peters et al., (2005, *JPO*)

The Jackson et al. (2008, *JPO*) Parameterization of Shear Instability

$$\frac{\partial}{\partial z}\left((\kappa+\nu_o)\frac{\partial Q}{\partial z}\right)+\kappa\|S\|^2-\kappa N^2-\left(c_N N+c_S\|S\|\right)Q=0 \quad \left(=\frac{DQ}{Dt}\right)$$

Term being reimplemented with new implicit expressions

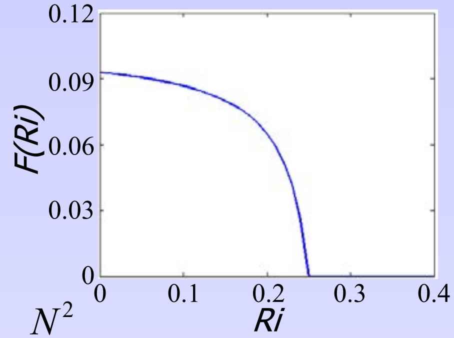
$$\frac{\kappa}{L_D^2}-\frac{\partial^2\kappa}{\partial z^2}=2\|S\|F(Ri)$$

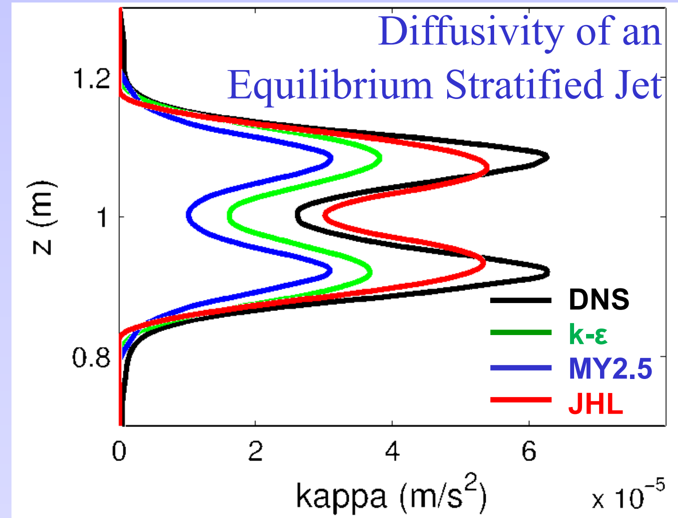
κ: Turbulent diapycnal diffusivity and viscosity [$m^2\ s^{-1}$]

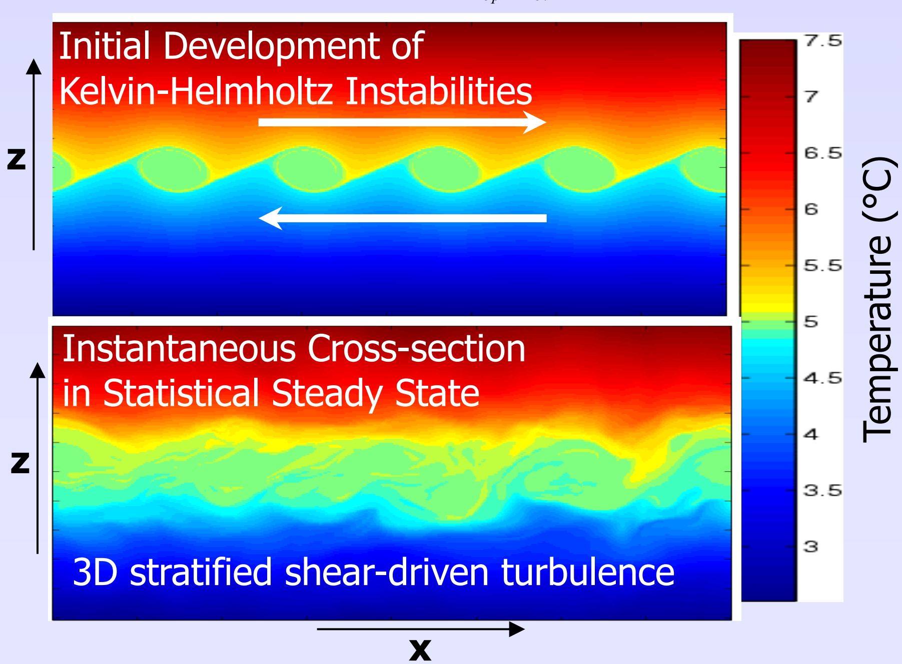
Q: TKE per unit mass [$m^2\ s^{-2}$]

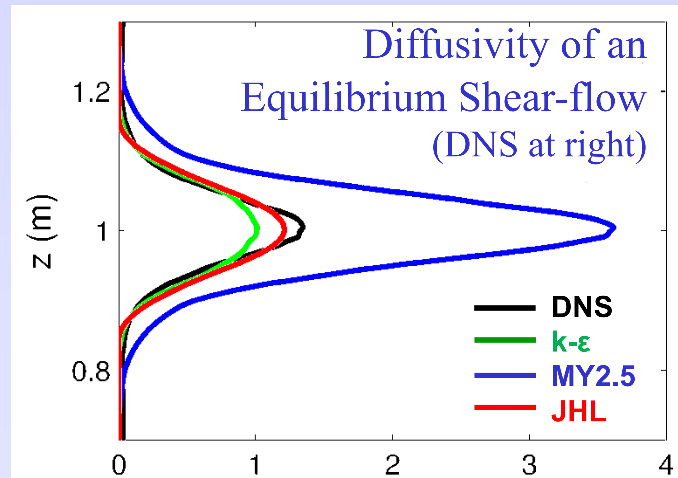
$$\|S\|^2=\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2 \qquad N^2=-\frac{g}{\rho_0}\frac{\partial \rho}{\partial z}$$

$$Ri=\frac{N^2}{S^2} \qquad L_D^{-2}=\frac{N^2}{\lambda^2 Q}+\frac{f^2}{Q}+\frac{\left(z_{Top}+z_{Bot}\right)^2}{z_{Top}^2\,z_{Bot}^2}\approx\frac{N^2}{\lambda^2 Q}$$









DNS – Results of 3-D DNS

k-ε – GOTM standard (~2008) k-ε closure (untuned)

MY – Mellor Yamada level 2.5 closure (untuned)

JHL –Jackson, et al, 2008 parameterization (tuned)

Take-away Messages

- Climate change will alter ocean mixing
  - Mixing stratified fluid takes energy
  - Energy sources for mixing will change, including the external and internal tides
  - Evolving stratification changes turbulent energetics
- The option to use the Integrated Implicit Energetics Approach is being applied to a wider range of mixing processes in MOM6
  - Bottom Boundary Layer ePBL - now available
  - Full column energetics-based mixing - underway
  - Revised Jackson stratified shear-mixing - planned