Quantum linear system algorithm for the two-dimensional radiative transfer equation

Gate-level resource estimation

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US CLIVAR Quantum Computing and Sensing for Weather and Climate Applications Workshop
August 4, 2026 Boulder, Colorado

Steady-state radiative transfer equation

$$\underbrace{\hat{\mathbf{\Omega}}\cdot\nabla I}_{\text{streaming}} = \underbrace{-\sigma_e I}_{\text{extinction}} + \underbrace{\sigma_a B(T)}_{\text{emission}} + \underbrace{\sigma_s \int_{4\pi} \Phi(\hat{\mathbf{\Omega}}' \to \hat{\mathbf{\Omega}}) I \, \mathrm{d}\Omega'}_{\text{scattering}}$$

$I(\boldsymbol{x}, \hat{\mathbf{\Omega}})$ radiance | $\hat{\mathbf{\Omega}}$ direction | $\sigma_e = \sigma_a + \sigma_s$ extinction / absorption / scattering coefficients | $B(T)$ Planck source | Φ phase function ($\int \Phi \, \mathrm{d}\Omega{=}1$)

Key properties

- **Linear in I**: coefficients fixed by the atmospheric state
- **Large**: unknowns span space and direction

Discretize with the discrete ordinates method

$\Rightarrow$ **large sparse linear system**

Modest (2013)

$$\mathbf{A}\mathbf{u} = \mathbf{f}, \quad \mathbf{A} \in \mathbb{R}^{N \times N}$$
$$N = N_{\text{cell}} \times N_{\text{ang}}$$

$\mathbf{u}$ radiances, one per (cell, direction)

$\mathbf{f}$ emission and boundary sources

Comparison with previous study

Time-marching approach

Time-dependent RTE

Quantum lattice Boltzmann method

N_t time steps

Igarashi et al. (2024)

Direct steady-state solve

Steady-state RTE

$\mathbf{A}\mathbf{u} = \mathbf{f}$

QLSA via QSVT

Steady state

This work

A direct steady-state formulation avoids the explicit time-stepping factor N_t.

QSVT matrix inversion

1. Block-encode ($\mathbf{A}$ itself is not unitary)

$$U_A = \begin{pmatrix} \mathbf{A}/\alpha & * \\ * & * \end{pmatrix} \Leftrightarrow \langle 0^a | U_A | 0^a \rangle = \frac{\mathbf{A}}{\alpha}$$

$\alpha \geq \|\mathbf{A}\|$ normalization factor | a ancillas

2. Transform every singular value

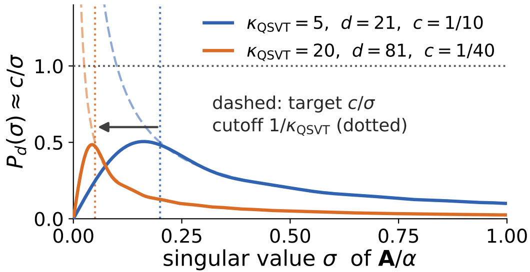
$$\frac{\mathbf{A}}{\alpha} = W \Sigma V^\dagger \xrightarrow{\text{QSVT}} W\, P_d(\Sigma)\, V^\dagger$$

W, V unitary | Σ singular values,
P_d degree-d polynomial

3. Take $P_d(\sigma) \approx c/\sigma$ (c constant factor)

$$V P_d(\Sigma) W^\dagger \approx c\, \alpha\, \mathbf{A}^{-1} \implies |u\rangle \propto \mathbf{A}^{-1} |f\rangle$$

QSVT and matrix inversion: Gilyén et al. (2019).



The degree is the cost:
the QSVT circuit calls U_A and $U_A^\dagger$ a total of d_{QSVT} times.

$$\boxed{d_{\mathrm{QSVT}} = O\left(\kappa_{\mathrm{QSVT}} \log \frac{\kappa_{\mathrm{QSVT}}}{\epsilon_{\mathrm{inv}}}\right), \quad \kappa_{\mathrm{QSVT}} \geq \kappa_{\mathrm{eff}} = \frac{1}{\sigma_{\min}}}$$

ϵ_{inv} target inversion error | $\sigma_{\min}$ smallest singular value of $\mathbf{A}/\alpha$

Drivers of the resource estimate

$$\text{solver core} \sim \underbrace{d_{\mathrm{QSVT}}}_{\text{set by } \kappa_{\mathrm{eff}}} \times \underbrace{N_T(U_A)}_{T \text{ gates per block encoding}}$$

$$\text{end-to-end} \sim \underbrace{N_W}_{\text{repetition count}} \times \text{solver core}$$

κ_{eff}

Measure it **numerically** for the discretized system

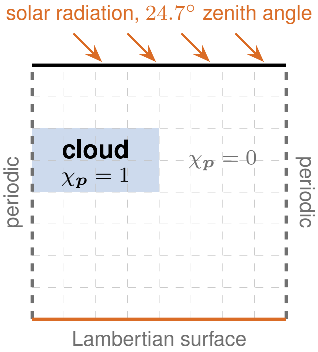
$N_T(U_A)$

Build the block encoding **explicitly** and count its gates

N_W

Determine it from the **observable overlap** and target precision

Problem setup



Optical coefficients: a binary cloud field

$$\sigma_{a/s,\boldsymbol{p}} = \sigma^{\mathrm{clr}}_{a/s} + \chi_{\boldsymbol{p}}\,\Delta\sigma_{a/s}, \qquad \chi_{\boldsymbol{p}} \in \{0, 1\}$$

$\boldsymbol{p}$ cell | $\chi_{\boldsymbol{p}}$ cloud flag | clr clear sky

shortwave (SW), $1.65\,\mu\mathrm{m}$	clear	in-cloud
absorption $\sigma_a\,[\mathrm{m}^{-1}]$	1.0×10^{-5}	1.55×10^{-4}
scattering $\sigma_s\,[\mathrm{m}^{-1}]$	0	2.49×10^{-2}

Phase function

clear sky: $\sigma_s = 0$, no Rayleigh scattering

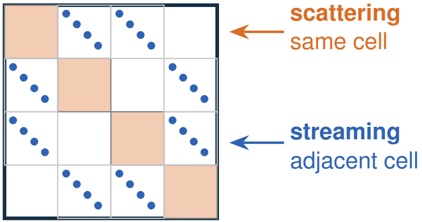
$$\text{in-cloud}: \quad \Phi^{\mathrm{HG}}(\cos\Theta) = \frac{1-g^2}{4\pi\,(1+g^2-2g\cos\Theta)^{3/2}}$$

Θ scattering angle | $g = 0.845$ asymmetry parameter, the Mie value for $r_{\mathrm{eff}} = 10\,\mu\mathrm{m}$ droplets.

Block-encoding strategy

Matrix structure

$$\mathbf{A}\mathbf{u} = \mathbf{f}$$



$\underbrace{}_{N_{\text{ang}}}$ $\mathbf{A}$: cells $\times$ directions

Sparse-access oracles

$$O_{\text{index}} : |i, k\rangle \mapsto |i, f(i,k)\rangle$$
$$O_{\text{value}} : |i, k, 0\rangle \mapsto \left(A_{i,f(i,k)}/\alpha\right) |i, k, 0\rangle + |\perp\rangle$$

i column | k-th nonzero in column i, row $f(i,k)$ | $|\perp\rangle$ rejected branch

O_{index}**: reversible index arithmetic**

- streaming: modular ± 1 **adder**
- scattering: controlled **SWAP**

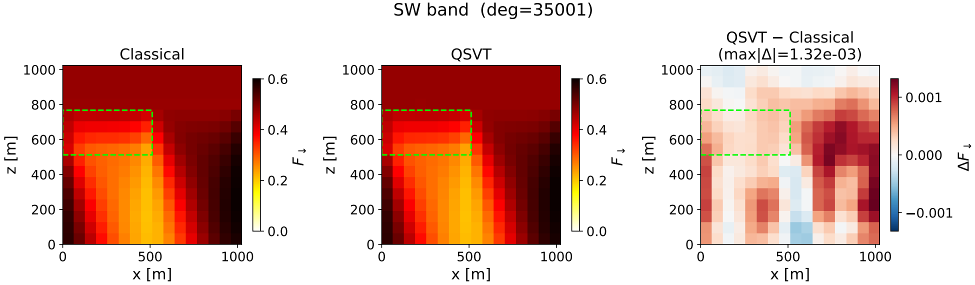
O_{value}**: controlled R_y rotations**

- controlled on direction pairs and the cloud flag
- no gates applied cell by cell

Downward-flux validation

Setup: 16×16 grid, $\Delta x = \Delta z = 64\,\mathrm{m}$, S_4 ordinate set ($N_{\mathrm{ang}} = 24$),

QSVT: $\kappa_{\mathrm{QSVT}} = 5000$, $d_{\mathrm{QSVT}} = 35001$



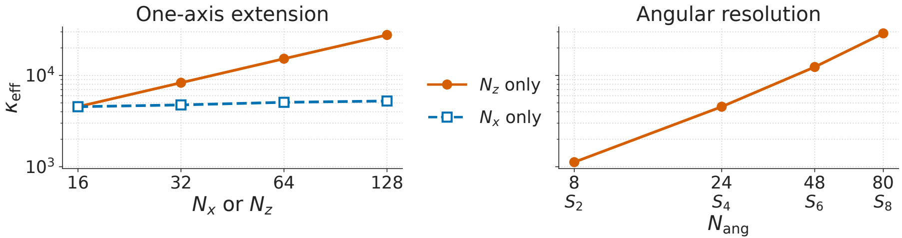
Circuits built with QURI Parts and simulated on the cuQuantum state-vector backend; a classical sparse direct solve is the reference.

0.16% radiance　0.13% downward flux

relative L^2 error at $d_{\mathrm{QSVT}} = 7\kappa_{\mathrm{QSVT}} + 1$

Effective condition number

Setup: $\Delta x = \Delta z = 64\,\mathrm{m}$ | left: S_4 fixed; vary N_x or N_z alone from a 16×16 grid | right: angle sweep on a 16×16 grid



Increase one axis: $16 \to 128$

N_z only: κ_{eff}: $\times 6.1$

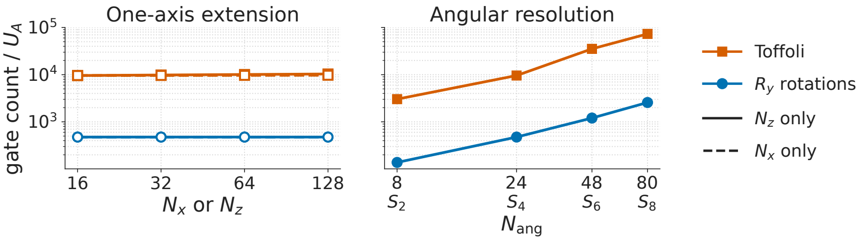
N_x only: κ_{eff}: $+15.5\%$

Increase angular resolution: $8 \to 80$

κ_{eff}: $\times 25.7$

U_A block-encoding cost

Setup: left: S_4 fixed; vary N_x or N_z alone from a 16×16 grid | right: angle sweep on a 16×16 grid



Gate counts from the QURI Parts `qsub` decomposition into H, X, CNOT, CZ, SWAP, Toffoli, and R_y; only Toffoli and R_y are plotted.

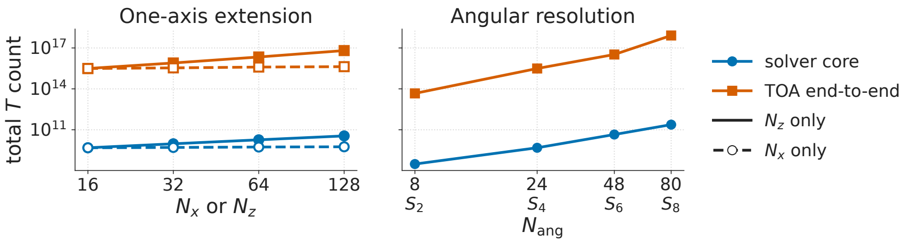
Increase one axis: $16 \to 128$
N_z: Toffoli $+7.8\%$, R_y unchanged
N_x: Toffoli $+1.5\%$, R_y unchanged

Increase angular resolution: $8 \to 80$
Toffoli $\times 24$, R_y $\times 19$

Solver core and end-to-end readout cost

Setup: $\Delta x = \Delta z = 64\,\mathrm{m}$ | left: S_4, vary N_x or N_z alone from 16×16 | right: S_2–S_8 on 16×16
$\kappa_{\mathrm{QSVT}} = 1.1\,\kappa_{\mathrm{eff}}$, $\epsilon_{\mathrm{inv}} = 5 \times 10^{-3}$, $\epsilon_{\mathrm{syn}} = 5 \times 10^{-3}$, top-of-atmosphere (TOA) flux readout to 1%

T **conversion:** Toffoli $= 7T$; rotation $= \lceil 10 + 4\log_2(N_{\mathrm{rot}}/\epsilon_{\mathrm{syn}}) \rceil T$ (N_{rot} rotations); preparation of $|f\rangle$ and of the observable state is ignored.



Amplitude estimation: Brassard et al. (2002).

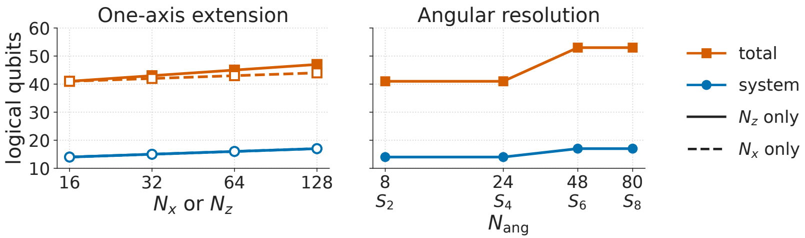
Increase one axis: $16 \to 128$
N_z: core $\times 7.4$, end-to-end $\times 20.6$
N_x: core $+18\%$, end-to-end $+36\%$

Increase angular resolution: $8 \to 80$
core $\times 776$, end-to-end $\times 1.76 \times 10^4$

Logical qubit requirements

Setup: $\Delta x = \Delta z = 64\,\mathrm{m}$, total: all registers in the synthesized U_A | system: (x, z, angle) registers

left: S_4 fixed; vary N_x or N_z alone from a 16×16 grid | right: angle sweep on a 16×16 grid



Increase one axis: $16 \to 128$

N_z: total/system $41/14 \to 47/17$

N_x: total/system $41/14 \to 44/17$

Increase angular resolution: $8 \to 80$

total/system: $41/14 \to 53/17$

Scaling summary and take-home messages

Grid extension	Cost driver	Solver core	TOA end-to-end
horizontal N_x	—	$\sim \log N_x$	$\sim \log N_x$
vertical N_z	$\kappa_{\text{eff}} \sim N_z^{0.87}$	$\sim N_z^{0.96} \approx N_z$	$\sim N_z^{1.46}$
directions N_{ang}	$\kappa_{\text{eff}} \sim N_{\text{ang}}^{1.40}$, $N_T(U_A) \sim N_{\text{ang}}^{1.39}$	$\sim N_{\text{ang}}^{2.89} \approx N_{\text{ang}}^3$	$\sim N_{\text{ang}}^{4.11} \approx N_{\text{ang}}^4$

Take-home messages for this simplified setting:

- **Extending the horizontal grid** is computationally inexpensive.
- Nevertheless, the **end-to-end cost** remains high: $\sim 10^{16}\ T$ gates even at these small problem sizes.

Acknowledgments

- This work was supported by JSPS KAKENHI Grant Number 26KJ0759.
- Computations were carried out on Supermicro ARS-111GL-DNHR-LCC and FUJITSU Server PRIMERGY CX2550 M7 (Miyabi) at the Joint Center for Advanced High Performance Computing (JCAHPC).
- We thank Dr. Yasunori Lee for providing the QSVT phase-angle synthesis used in this work.

A manuscript based on this work is in preparation.

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