Fuzzy Differentiation Tools

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Background

Uncertainty can be epistemic

Parameter uncertainty when repeating a measurement is difficult or impossible

We have some qualitative information, expert opinion only

Uncertainty due to the natural variability

Uncertainties due to the modeling process

Background

Classical Analysis

- Real numbers
- Linear space structure

- Metric structure -

Euclidean distance

Interval/Fuzzy Analysis

- Intervals/Fuzzy numbers
- Interval/Fuzzy aritmetics

- Metric structure -

Hausdorff distance between intervals/ fuzzy numbers

Background

- Propagation of Uncertainty-

Statistical

Random variables

Stochastic Differential Equations

Epistemic

Intervals/ Fuzzy Numbers

Interval/ Fuzzy Differential Equations

Fuzzy Differential Equations

There are several **interpretations** of a FDE:

•The interpretation based on the Hukuhara derivative (Puri, Ralescu) (the solution has increasing length of its support)

• The interpretation based on Zadeh's extension (Buckley, Feuring) (the derivative is not defined)

• Interpretation with differential inclusions (Hüllermeyer) (the derivative is not defined)

• Interpretation based on generalized differentiability (B-Gal)

The main **difficulty** in the study of FDEs:

•Peano type existence results are not proved yet.

Nieto claimed a proof, but very recently, Dontchev has found a flaw in the proof of Nieto. Namely since R_F is not locally compact a closed ball is not necessarily compact so we might have bounded sequence without a convergent subsequence.

Hukuhara derivative (fuzzy case)

Hukuhara difference. Let x,y be fuzzy numbers.
The H-difference of x and y is $z = x \ominus y$ if $y + z = x.$

Hukuhara derivative. The function $f:(a,b) \to R_F$ is said to be differentiable if the following limits exist together with the Hukuhara differences

$$\lim_{h\to 0^+} \frac{f(x_0+h) \ominus f(x_0)}{h} = \lim_{h\to 0^+} \frac{f(x_0) \ominus f(x_0-h)}{h} = f'(x_0),$$

Mainly Existence and Uniqueness results were obtained (Puri-Ralescu, Seikkala, Wu Congxin, Kaleva).
The main shortcoming of this interpretation is that the solution of a FIVP has increasing length of the support.

Generalized differentiability

Strongly Generalized Differentiability (B-Gal 2005)

$f:(a,b)\to R_F$ is strongly generalized differentiable at x_0 if

$$(i)\ \lim_{h\to 0^+}\frac{f(x_0+h)\ominus f(x_0)}{h}=\lim_{h\to 0^+}\frac{f(x_0)\ominus f(x_0-h)}{h}=f'(x_0),$$

$$(ii)\ \lim_{h\to 0^+}\frac{f(x_0)\ominus f(x_0+h)}{-h}=\lim_{h\to 0^+}\frac{f(x_0-h)\ominus f(x_0)}{-h}=f'(x_0),$$

$$(iii)\ \lim_{h\to 0^+}\frac{f(x_0+h)\ominus f(x_0)}{h}=\lim_{h\to 0^+}\frac{f(x_0-h)\ominus f(x_0)}{-h}=f'(x_0),$$

$$(iv)\ \lim_{h\to 0^+}\frac{f(x_0)\ominus f(x_0+h)}{-h}=\lim_{h\to 0^+}\frac{f(x_0)\ominus f(x_0-h)}{h}=f'(x_0),$$

Generalized differentiability

Generalized Hukuhara Differentiability (Stefanini-B09)

$$\lim_{h\to 0}\frac{f(x_0+h)\ominus_{gH} f(x_0)}{h}=\lim_{h\to 0}\frac{f(x_0)\ominus_{gH} f(x_0-h)}{h}=f'(x_0),$$

There are two cases and possibly switches between them

$$\text{(i.)}\qquad \left[f'(x_0)\right]_\alpha=[\left(f_\alpha^-\right)'(x_0),\left(f_\alpha^+\right)'(x_0)],\forall\alpha\in[0,1]$$

$$\text{(ii.)}\qquad \left[f'(x_0)\right]_\alpha=[\left(f_\alpha^+\right)'(x_0),\left(f_\alpha^-\right)'(x_0)],\forall\alpha\in[0,1].$$

Remarks. Weakly generalized differentiability is equivalent with generalized Hukuhara differentiability.

Strongly generalized differentiability is slightly stronger.

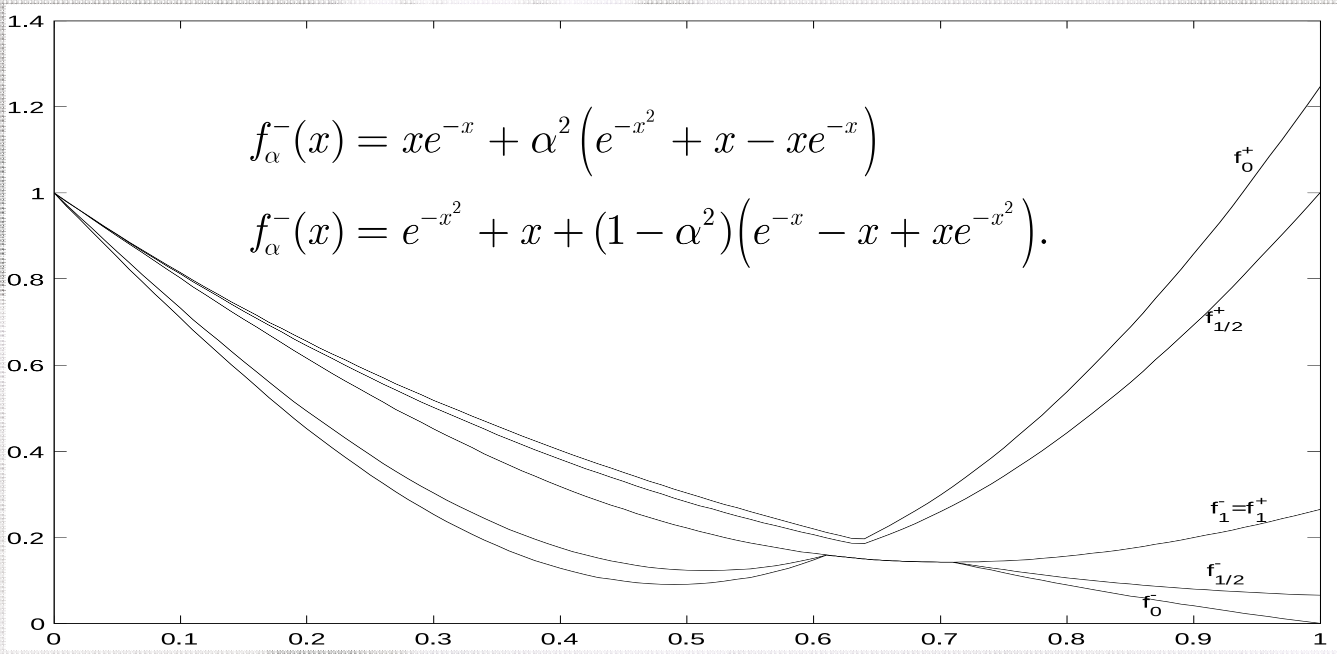
It allows only isolated switching points.

Generalized differentiability

Generalized Fuzzy Differentiability (g-differentiability)

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)\ominus_g f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0)\ominus_g f(x_0-h)}{h}$$

This concept is more general than the previous concepts.



Existence and uniqueness of two solutions

Theorem. (B-Gal 2005,2010) Let us consider the FIVP

$$y' = F(x, y),\ y(x_0) = y_0$$

(i) F is continuous, Lipschitz in the second argument

(ii) $F_\alpha^\pm$ have bounded partial derivatives w.r.t. α

(iii) $x_0^-{}'(\alpha) \le c_2 < 0 < c_1 \le x_0^+{}'(\alpha)$ and

a) $x_0^-(1) < x_0^+(1)$ or

b) if $x_0^-(1) = x_0^+(1)$ then the core $[F(t,u)]^1$ consists in exactly one element whenever $[u]^1$ consists in exactly one element.

Then, the fuzzy initial value problem (FIVP) has two unique solutions locally.

Corollary. If F is a triangular or trapezoidal valued function and $x_0 = (a,b,c,d)$ is a triangular or trapezoidal number with $a < b \le c < d$ then there exist locally two unique solutions.

Characterization, Fuzzy Differential Equations

Theorem. (B, 2006, B-Gal, 2010) Let us consider the FIVP

$$y' = F(x, y),\ y(x_0) = y_0$$

With $[F(t,x)]^\alpha = [F_\alpha^-(t, x_\alpha^-, x_\alpha^+), F_\alpha^+(t, x_\alpha^-, x_\alpha^+)], \forall \alpha \in [0,1].$

(i) If $F_\alpha^\pm(t, x_\alpha^-, x_\alpha^+)$ are equicontinuous and uniformly Lipshchitz in their second and third arguments.

(ii) $F_\alpha^\pm$ have bounded partial derivatives w.r.t $\alpha \in [0,1].$

(iii) $x_0^-{}'(\alpha) \le c_2 < 0 < c_1 \le x_0^+{}'(\alpha)$ and

a) $x_0^-(1) < x_0^+(1)$ or

b) if $x_0^-(1) = x_0^+(1)$ then the core $[F(t,x)]^1$ consists in exactly one element whenever $[x]^1$ consists in exactly one element.

Characterization, Fuzzy Differential Equations

Then the FIVP $y' = F(x,y),\ y(x_0) = y_0$ is equivalent with
The union of the ODEs:

(a)

$$\begin{cases} \left(x_\alpha^-\right)'(t) = F_\alpha^-\left(t, x_\alpha^-(t), x_\alpha^+(t)\right) \\ \left(x_\alpha^+\right)'(t) = F_\alpha^+\left(t, x_\alpha^-(t), x_\alpha^+(t)\right), \alpha \in [0,1] \\ x_\alpha^-(t_0) = \left(x_0\right)_\alpha^-, x_\alpha^+(t_0) = \left(x_0\right)_\alpha^+ \end{cases}$$

(b)

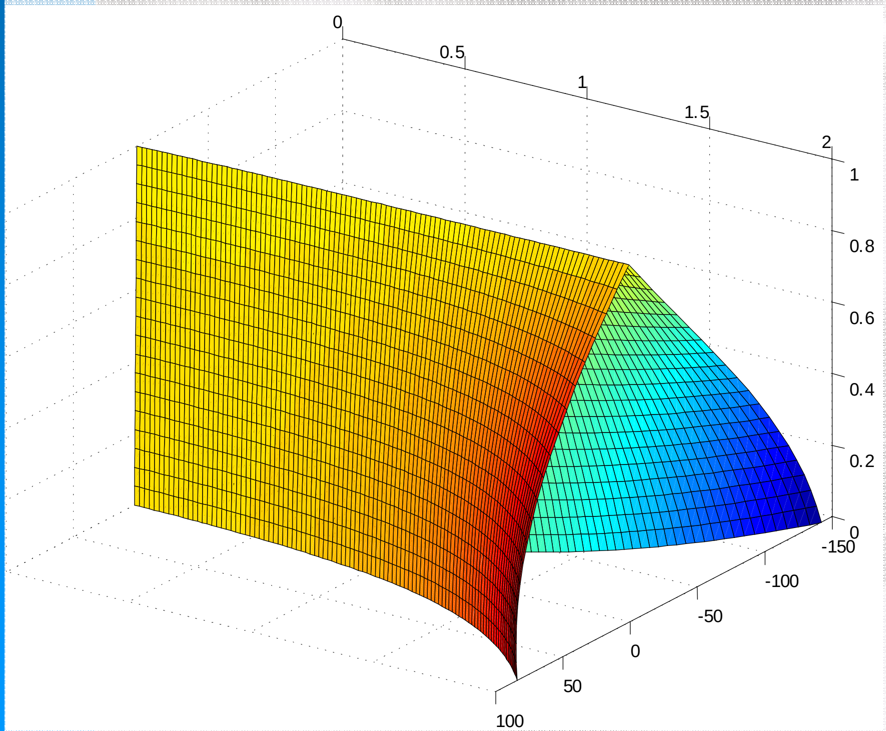
$$\begin{cases} \left(x_\alpha^-\right)'(t) = F_\alpha^+\left(t, x_\alpha^-(t), x_\alpha^+(t)\right) \\ \left(x_\alpha^+\right)'(t) = F_\alpha^-\left(t, x_\alpha^-(t), x_\alpha^+(t)\right), \alpha \in [0,1] \\ x_\alpha^-(t_0) = \left(x_0\right)_\alpha^-, x_\alpha^+(t_0) = \left(x_0\right)_\alpha^+ \end{cases}$$

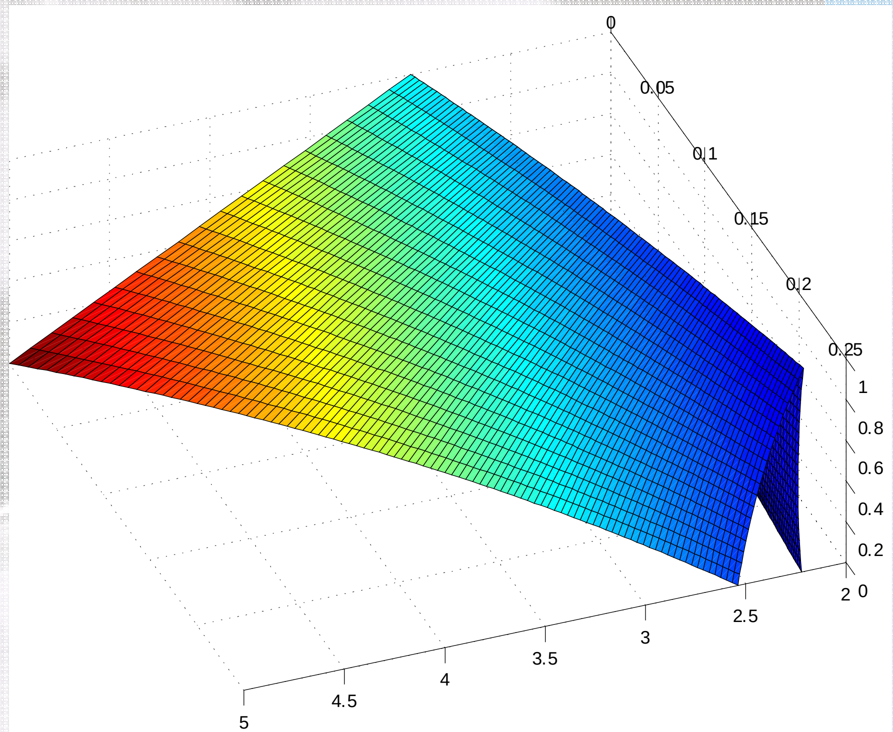
Examples, Fuzzy Differential Equations

Let us consider the following general linear equation

$$x' = a(t) \cdot x + b(t),\ \ x(t_0) = x_0,$$

with $x_0 \in R_F,\ a, b\ \ : [t_0, t_0 + p] \to R_F$

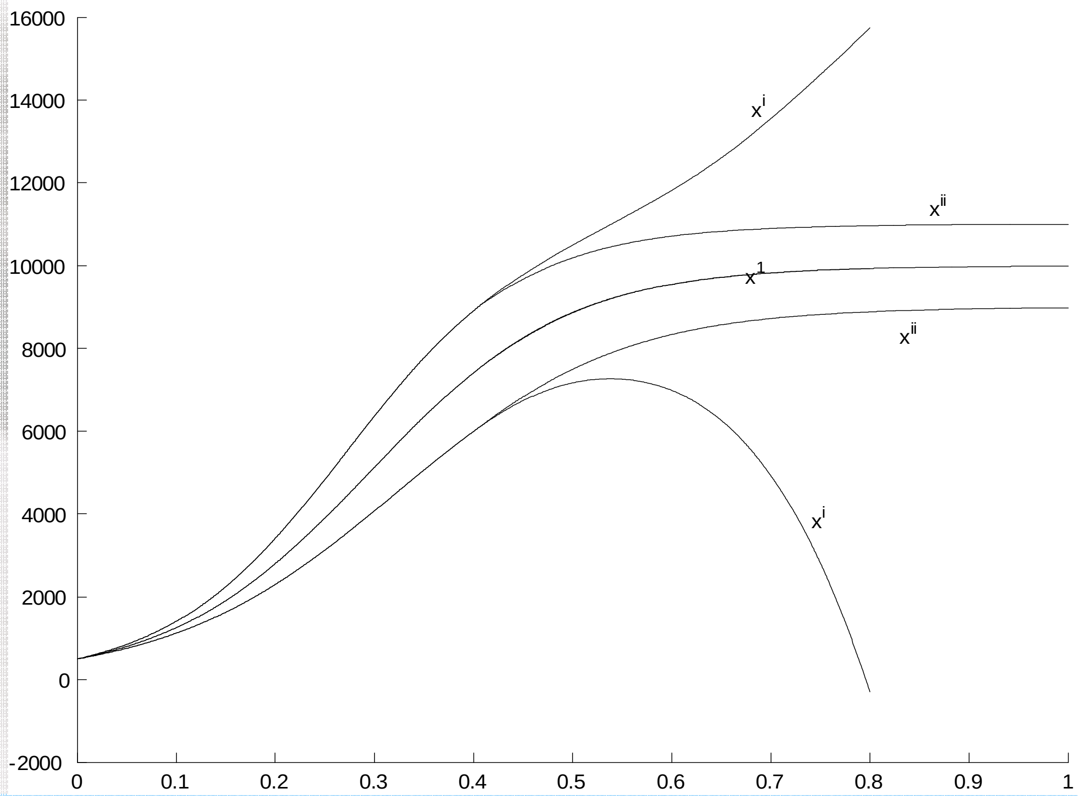




Examples, Fuzzy Differential Equations

Recently, Nieto and Rodriguez-Lopez considered the fuzzy logistic equation

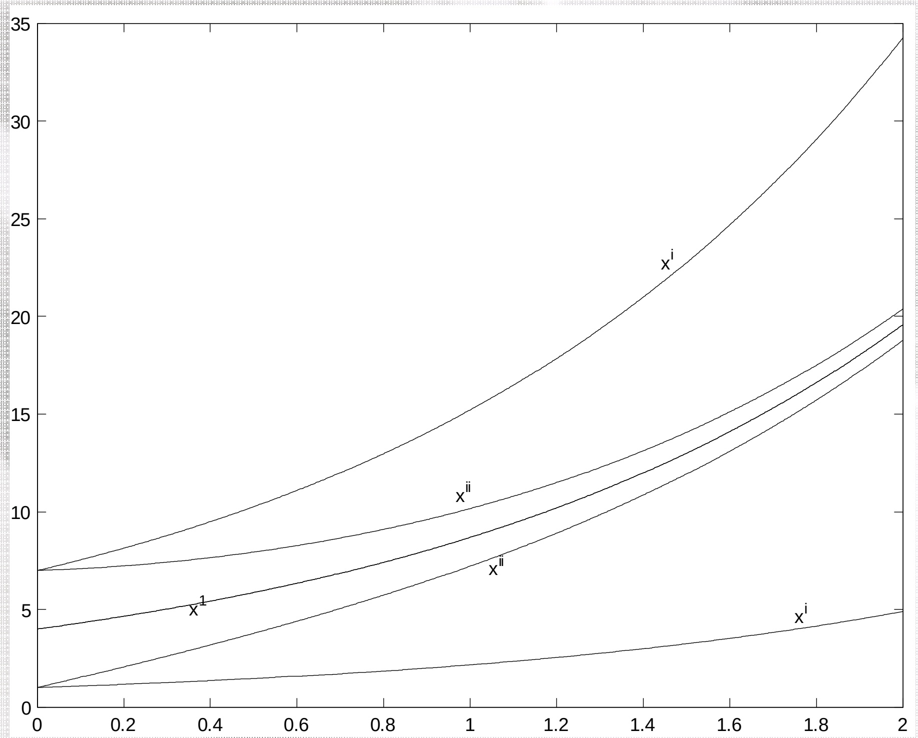
$$x' = ax(K \ominus_{gH} x), \; x(0) = x_0$$



Examples, Fuzzy Differential Equations

Fuzzy differential equations with deviated argument were recently considered as the following fuzzy pantograph equation

$$x'(t) = ax(t) + bx(qt), x(0) = x_0$$



Further Research

- Modeling under epistemic uncertainty becomes a more and more important issue in safety analysis.

- Propagation of uncertainties in climate models can be modeled using Fuzzy Differential Equations.

- To solve Fuzzy Differential Equations in climate models we will combine our characterization results with Automatic Differentiation tools.

Thank You